Cyber-Physical System Security, Optimal Control, and Consensus Protocols for Nonlinear Stochastic Systems

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To my parents, Jianping Jin and Yufen Tian

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Summary

Recent technological advances in communications and computation have spurred a broad interest in control law architectures involving the monitoring, coordination, integration, and operation of sensing, computing, and communication components that tightly interact with the physical processes that they control. These systems are known as cyber-physical systems and due to their use of open computation and communication platform architectures, controlled cyber-physical systems are vulnerable to adversarial attacks. In this thesis, we propose a novel adaptive control architecture for addressing security and safety in cyber-physical systems. Specifically, we develop an adaptive controller that guarantees uniform ultimate boundedness of the closed-loop dynamical system in the face of adversarial sensor and actuator attacks that are time-varying and partial asymptotic stability when the sensor and actuator attacks are time-invariant. Next, we build on this framework to develop an adaptive control algorithm for addressing security for a class of networked vehicles that comprise n human-driven vehicles sharing kinematic data and an autonomous vehicle in the aft of the vehicle formation receiving data from the preceding vehicles by wireless vehicle-to-vehicle communication devices. Specifically, we develop an adaptive controller for mitigating time-invariant, state-dependent adversarial sensor and actuator attacks while guaranteeing uniform ultimate boundedness of the closed-loop networked system.

Next, we propose a novel adaptive control architecture for addressing security and safety in cyber-physical systems subject to exogenous disturbances. Specifically, we develop an adaptive controller for time-invariant, state-dependent adversarial sensor and actuator attacks in the face of stochastic exogenous disturbances modeled as Markov processes. We

show that the proposed controller guarantees uniform ultimate boundedness of the closed-loop dynamical system in a mean-square sense. We further discuss the practicality of the proposed approach and apply the proposed framework to the lateral directional dynamics of an aircraft to illustrate the efficacy of the adaptive control architecture.

Then, we address networked multiagent systems subject to stochastic exogenous disturbances with compromised sensor and actuators. First, for a class of linear leader-follower multiagent systems, we develop a new structure of the neighborhood synchronization error for the control design protocol of each follower. The proposed control algorithm addresses time-varying multiplicative sensor attacks on the leader state measurements. In addition, the framework addresses time-varying multiplicative actuator attacks on the followers that do not have a communication link with the leader and additive actuator attacks on all follower agents in the network. The proposed adaptive controller guarantees uniform ultimate boundedness of the state tracking error for each agent in a mean-square sense.

Next, we extend the approach to develop a distributed robust adaptive control architecture that can foil malicious sensor and actuator attacks in the face of exogenous stochastic disturbances and follower agent model uncertainties. Specifically, for a class of linear multiagent uncertain systems with an undirected communication graph topology we develop a neighborhood synchronization error for the distributed robust adaptive control protocol design of each follower to account for actuator and sensor attacks on the leader state as well as all of the follower agents in the network. The proposed robust adaptive controller guarantees uniform ultimate boundedness in probability of the state tracking error for each follower agent in a mean-square sense. To show the efficacy of our adaptive control architecture, we provide several numerical illustrative examples involving the lateral directional dynamics of an aircraft group of agents subject to state-dependent atmospheric drag disturbances, sensor and actuator attacks, and follower agent model uncertainties. Finally, the framework is extended to address output feedback architectures for leader-follower multiagent systems with stochastic disturbances and sensor and actuator attacks.

We then turn our attention to the development of an energy-based static and dynamic control framework for stochastic port-controlled Hamiltonian systems. In particular, we obtain constructive sufficient conditions for stochastic feedback stabilization that provide a shaped energy function for the closed-loop system while preserving a Hamiltonian structure at the closed-loop level. In the dynamic control case, energy shaping is achieved by combining the physical energy of the plant and the emulated energy of the controller. Several numerical examples are presented that demonstrate the efficacy of the proposed passivity-based stochastic control framework.

Building on a stochastic optimal control framework, we derive stability margins for optimal and inverse optimal stochastic feedback regulators. Specifically, gain, sector, and disk margin guarantees are obtained for nonlinear stochastic dynamical systems controlled by nonlinear optimal and inverse optimal Hamilton-Jacobi-Bellman controllers that minimize a nonlinear-nonquadratic performance criterion with cross-weighting terms. Furthermore, using the newly developed notion of stochastic dissipativity we derive a return difference inequality to provide connections between stochastic dissipativity and optimality of nonlinear controllers for stochastic dynamical systems. In particular, using extended Kalman-Yakubovich-Popov conditions characterizing stochastic dissipativity we show that our optimal feedback control law satisfies a return difference inequality predicated on the infinitesimal generator of a controlled Markov diffusion process if and only if the controller is stochastically dissipative with respect to a specific quadratic supply rate.

A constructive finite time stabilizing feedback control law is derived next for stochastic dynamical systems driven by Wiener processes based on the existence of a stochastic control Lyapunov function. In addition, we present necessary and sufficient conditions for continuity of such controllers. Moreover, using stochastic control Lyapunov functions, we construct a universal inverse optimal feedback control law for nonlinear stochastic dynamical systems that possesses guaranteed gain and sector margins.

Finally, we focus on semistability and finite time semistability analysis and synthesis of stochastic dynamical systems having a continuum of equilibria. Stochastic semistability is the property whereby the solutions of a stochastic dynamical system almost surely converge to Lyapunov stable in probability equilibrium points determined by the system initial conditions. We extend the theories of semistability and finite-time semistability for deterministic dynamical systems to develop a rigorous framework for stochastic semistability and stochastic finite-time semistability. Specifically, Lyapunov and converse Lyapunov theorems for stochastic semistability are developed for dynamical systems driven by Markov diffusion processes. These results are then used to develop a general framework for designing semistable consensus protocols for dynamical networks in the face of stochastic communication uncertainty for achieving multiagent coordination tasks in finite time. The proposed controller architectures involve the exchange of generalized charge or energy state information between agents guaranteeing that the closed-loop dynamical network is stochastically semistable to an equipartitioned equilibrium representing a state of almost sure consensus consistent with basic thermodynamic principles.

Chapter 1

Introduction

1.1. Motivation and Goals

Recent technological advances in communications and computation have spurred a broad interest in control law architectures involving the monitoring, coordination, integration, and operation of sensing, computing, and communication components that tightly interact with the physical processes that they control. These systems are known as *cyber-physical systems* (see [2] and the references therein) and even though they are transforming the way we are interacting with the physical world, they introduce several grand research challenges. In particular, due to the use of open computation and communication platform architectures, controlled cyber-physical systems are vulnerable to adversarial attacks. Cyber attacks can severely compromise system stability, performance, and integrity. Specifically, malicious attacks in feedback control systems can compromise sensor measurements as well as actuator commands to severely degrade closed-loop system performance and integrity. The pervasive security and safety challenges underlying cyber-physical systems place additional burdens on standard adaptive control methods.

Depending on the available resources of the attacker and the knowledge of the system, different malicious attacks can be injected into cyber-physical systems [137]. These available resources are generally categorized as disclosure and disruption resources. Disclosure resources enable an attacker to gather sensitive information about the system, such as sensor measurements and control input commands, whereas disruption resources enable the attacker to affect and alter the system operation by violating data integrity. Such attacks can include eavesdropping, denial-of-service, and false data injection.

In contrast to classical estimation and control problems, wherein physical system variables cannot be measured directly due to sensor noise and are typically assumed to fluctuate about their true value, controlled systems with measurement and actuation devices that are hijacked and controlled by an adversarial entity that actively engages to maximally degrade system information and control require new and novel control algorithms to recover system performance. Cyber attacks are continuously becoming more sophisticated and intelligent, and hence, it is vital to develop algorithms that can suppress their effects on cyber-physical systems [69]. Cyber-physical security involving information security and detection in adversarial environments have thus been considered in the literature [15, 34, 48, 61, 71, 72, 89, 107, 125, 129, 138, 147].

Early approaches focused on classical fault detection, isolation, and recovery schemes (see, for example, [15, 61, 89] and the references therein). Specifically, in [107] an attack detection and identification algorithm is developed based on classical fault detection, isolation, and recovery schemes [15, 89]. In this approach, residual signals are generated based on the difference between the sensor measurements and the system output. These signals are then used to detect and identify attacks on the system. However, in practice it is difficult to generate a residual signal for each potential attack mode and, as the number of attack modes increases, this approach becomes impractical. Furthermore, a key assumption of the classical fault detection, isolation, and recovery schemes is that all dynamical system signals remain bounded during the fault detection process, which is not a valid assumption; especially if the adversarial attacks are state dependent.

More recently, an analysis, design, and evaluation framework of resilient monitoring systems for sensor networks that degrade gracefully under malicious sensor attacks was presented in [43], whereas [146] considers an information flow analyses framework to support passive and active detection of adversaries in cyber-physical systems. The authors in [85] present a set-theoretic control framework for cyber-physical systems to derive an anomaly detector module, wherein false data injection attacks are modeled as additive signals to sensor measurements and actuator commands. In [138], a game theoretic approach is developed to estimate a binary random variable predicated on sensor measurements that have been corrupted by a cyber-attacker. A detection problem for mitigating deception attacks in cyber-physical systems is presented in [75], whereas a security problem for cyber-physical systems is presented in [80] and a remote state estimation algorithm using multiple sensors is proposed.

In [48], a dynamic game framework is proposed wherein an attacker actively and optimally perturbs the controller by using a finite-number of jamming actions over a finite-horizon. However, their results are limited to scalar systems. In [34], the authors model adversarial attacks on actuators and sensors as exogenous disturbances. However, the proposed control framework cannot address cases wherein more than half of the sensors are compromised and the set of attack nodes are time varying. In [156], a malicious deception sensor attack model for cyber-physical systems is considered, where it is assumed that the attack signal is added to the states of the system. Then using an adaptive control framework the effect of the sensor attacks is suppressed. Building on the framework of [156], in [67] both malicious deception sensor and actuator attacks are considered.

Data injection attacks on power grid models based on steady-state operation are presented in [71, 72, 86, 129], where an attacker manipulates the system state estimator of the smart grid. In [105], a cyber-physical security framework for an energy management system is proposed and the physical correlations between data points are identified to detect outliers. In the presence of an outlier, the feedback loop is then closed using an estimated value of the sensor measurements. Finally, the authors in [3, 33] develop an adaptive control algorithm for mitigating the effects of sensor uncertainties in networked multiagent systems.

Multiagent systems comprise an important subclass of cyber-physical systems that involve communication and collaboration between interacting agents that locally exchange information. In particular, leader-follower consensus has a wide application in areas such as surveillance, formation control, and search and rescue. In such systems, the system state information of different agents is exchanged through communication channels represented by a given graph communication topology, and local actuators of each agent utilize the information received from its neighbors for the control design protocol. For the leader-follower consensus problems, most of the results in the literature assume that at least a subset of the followers have access to the *exact* leader state information [29, 108, 131, 145, 154, 158, 159]. However, in realistic situations, the leader state information measured or received by the follower agents may be corrupted due to an attack on the communication channel. Consequently, each follower, which has a communication link with the leader, may measure or receive erroneous leader state information. In other words, every follower agent may have inexact state information for the leader.

An important application area of multiagent cyber-physical systems is in networked autonomous transportation systems. The problem of control design of vehicle platoons has attracted considerable attention among researchers in the field of control, optimization, and communication [23, 74, 92, 117, 130]. Given the increasing numbers of transportation congestion and accidents world-wide, extensive research efforts have been devoted to increasing the adaptation, autonomy, connectivity, safety, and reliability of vehicular platoon control systems. Connected networks of vehicles often involve distributed decision-making for coordination involving information flow enabling enhanced operational effectiveness via cooperation. It is evident that as the technology and complexity of autonomous vehicles evolves, several grand research challenges need to be addressed. These include securing the autonomous vehicle from malicious cyber attacks that might increase engine revolutions per minute, disabling a cylinder or even disengaging the engine completely, activating airbags while driving to obscure vision, tampering with the breaking system causing a skid or preventing the braking system from being engaged when driving; setting the display to an erroneous speed so that the driver is unaware they are violating speed limits, and instigating a malfunction in the vehicle's position system.

The design and implementation of secure control framework for a connected autonomous transportation systems is a nontrivial task involving the consideration and operation of computing and communication components (see [2] and the references therein) interacting with the physical, cyber, and human-in-the-loop processes. Even though adaptive control can be used to address autonomous networked systems, the pervasive security and safety challenges underlying connected autonomous transportation systems place additional burdens on standard adaptive control methods. Specifically, although adaptive learning architectures have been used in numerous applications to achieve stability and improve system performance, their standard architectures are *not* designed to address adversarial attacks.

In numerous applications where dynamical system models are used to describe the behavior of natural and engineering systems, stochastic components and random disturbances are typically incorporated into the models. The stochastic aspects of the models are used to quantify system uncertainty and system disturbances as well as the dynamic relationships of sequences of random events between system-environment interactions. In the recent papers [111], [114], the authors extend classical deterministic dissipativity theory [148] to nonlinear stochastic dynamical systems using basic input-output and state properties. Specifically, a stochastic version of dissipativity theory using both an input-output as well as a state dissipation inequality in expectation for controlled Markov diffusion processes is presented.

Dissipativity theory and in particular passivity-based control frameworks for deterministic port-controlled Hamiltonian systems using energy shaping have been developed in the literature. Specifically, the authors in [100–102] develop a control design methodology that achieves stabilization via system passivation. In light of the fact that energy notions involving conservation, dissipation, and transport of energy also arise naturally for dissipative diffusion processes, it seems natural that dissipativity theory can play a key role in the control design of stochastic dynamical systems. Specifically, stochastic dissipativity and passivity theory can be used to design feedback controllers that add dissipation and guarantee stability robustness in probability allowing stochastic stabilization to be understood in physical terms.

1.2. Outline of the Proposed Research

The contents of this dissertation can be segmented in three parts; namely, adaptive control for cyber-physical systems, stochastic optimal control, and network consensus control with communication uncertainty. More specifically, in Chapter 2, we propose a novel adaptive control architecture for addressing security and safety in cyber-physical systems, and apply our framework to autonomous vehicle platoon systems. In Chapter 3, we propose a novel adaptive control architecture for addressing security and safety in cyber-physical systems subject to exogenous disturbances. Specifically, we develop an adaptive controller for time-invariant, state-dependent adversarial sensor and actuator attacks in the face of stochastic exogenous disturbances modeled as Markov processes. Then, in Chapter 4, we develop novel state and output feedback distributed adaptive control architectures for addressing networked multiagent systems subject to stochastic exogenous disturbances with compromised sensor and actuators.

In Chapter 5, we develop an energy-based static and dynamic control framework for stochastic port-controlled Hamiltonian systems. In particular, we obtain constructive sufficient conditions for stochastic feedback stabilization that provide a shaped energy function for the closed-loop system while preserving a Hamiltonian structure at the closed-loop level. In Chapter 6, we derive stability margins for optimal and inverse optimal stochastic feedback regulators. Specifically, gain, sector, and disk margin guarantees are obtained for nonlinear stochastic dynamical systems controlled by nonlinear optimal and inverse optimal Hamilton-Jacobi-Bellman controllers that minimize a nonlinear-nonquadratic performance criterion with cross-weighting terms. Furthermore, using the newly developed notion of stochastic dissipativity we derive a return difference inequality to provide connections between stochastic dissipativity and optimality of nonlinear controllers for stochastic dynamical systems. In Chapter 7, we derive a constructive finite time stabilizing feedback control law for stochastic dynamical systems driven by Wiener processes based on the existence of a stochastic control Lyapunov function. In addition, we present necessary and sufficient conditions for continuity of such controllers. Moreover, using stochastic control Lyapunov functions, we construct a universal inverse optimal feedback control law for nonlinear stochastic dynamical systems that possesses guaranteed gain and sector margins.

In Chapter 8, we turn our attention to network systems. To address the problem of consensus control in stochastic networks we first extend the theories of semistability and finite-time semistability for deterministic dynamical systems to develop a rigorous framework for stochastic semistability and stochastic finite-time semistability. Specifically, Lyapunov and converse Lyapunov theorems for stochastic semistability are developed for dynamical systems driven by Markov diffusion processes.

Finally, in Chapter 9, we discuss ongoing research and future extensions of the proposed research.

Chapter 2

Adaptive Control Architectures for Deterministic Systems with Sensor and Actuator Attacks

2.1. Introduction

Cyber-physical system security involving information security and detection in adversarial environments have been considered in the literature [11, 15, 34, 48, 61, 71, 72, 89, 90, 104, 106, 107, 125, 129, 137, 138, 147], with early approaches focusing on classical fault detection, isolation, and recovery schemes (see, for example, [15, 61, 89] and the references therein). In the first part of this chapter, we build on the solid foundation of adaptive control theory to develop new adaptive control architectures that can foil malicious sensor and actuator attacks. Specifically, we develop an adaptive controller for mitigating time-varying and time-invariant, state-dependent sensor and actuator attacks. We show that the proposed controller guarantees uniform ultimate boundedness of the closed-loop dynamical system when the adversarial sensor and actuator attacks are time-varying and partial asymptotic stability when the sensor and actuator attacks are time-invariant. Finally, we discuss the practicality of the proposed approach and provide a numerical example involving the lateral directional dynamics of an aircraft to illustrate the efficacy of the proposed adaptive control architecture.

In the second part of this chapter, we build on the adaptive control framework of [66] to develop an adaptive controller for a team of connected vehicles subject to time-invariant, state-dependent sensor and actuator attacks. The proposed controller guarantees uniform ultimate boundedness of the closed-loop networked system. The adaptive controller is composed of two components, namely a nominal controller and an additive corrective signal. It is assumed that the nominal controller has been already designed and implemented to achieve a desired closed-loop nominal performance. Using the nominal controller, an additive adaptive corrective signal is designed and added to the output of the nominal controller in order to suppress the effects of the sensor and actuator attacks. Thus, the proposed controller is modular in the sense that there is no need to redesign the nominal controller in the proposed framework; only the adaptive corrective signal is designed using the available information from the nominal controller and the system.

The notation used in this chapter is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, $(\cdot)^{\mathrm{T}}$ denotes the transpose operator, $(\cdot)^{-1}$ denotes the inverse operator, $\det(\cdot)$ denotes the determinant operator, $\|\cdot\|_1$ denotes the absolute sum norm, $\|\cdot\|_2$ denotes the Euclidian norm, and $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius matrix norm. Furthermore, we write $\lambda_{\min}(A)$ (resp., $\lambda_{\max}(A)$) for the minimum (resp., maximum) eigenvalue of the matrix A, $\mathrm{spec}(A)$ for the spectrum of the matrix A including multiplicity, and $\underline{x}$ (resp., $\overline{x}$) for the lower bound (resp., upper bound) of a bounded signal; that is, for $x(t) \in \mathbb{R}^n$, $t \geq 0$, $\underline{x} \leq \|x(t)\|_2$, $t \geq 0$ (resp., $\|x(t)\|_2 \leq \overline{x}$, $t \geq 0$), and for $X(t) \in \mathbb{R}^{p\times m}$, $t \geq 0$, $\underline{x} \leq \|X(t)\|_{\mathrm{F}}$, $t \geq 0$ (resp., $\|X(t)\|_{\mathrm{F}} \leq \overline{x}$, $t \geq 0$). Finally, for $y \in \mathbb{R}^n$, y_i denotes the ith component of y.

2.2. Adaptive Control Architecture for Mitigating Sensor and Actuator Attacks in Cyber-Physical Systems

We consider linear dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, and $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are known system matrices. We assume that the pair (A, B)

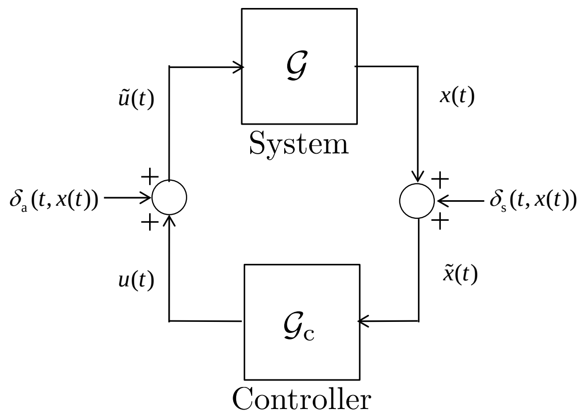


Figure 2.2.1: Closed-loop dynamical system in the presence of sensor and actuator attacks.

is controllable and the control input $u(\cdot)$ is restricted to the class of admissible controls consisting of measurable functions such that $u(t) \in \mathbb{R}^m$, $t \geq 0$. In addition, we assume that the compromised system state

$$\tilde{x}(t) = x(t) + \delta_{\mathrm{s}}(t, x(t)), \quad t \geq 0, \tag{2.2}$$

is available for feedback, where $\tilde{x}(t) \in \mathbb{R}^n$, $t \geq 0$, and $\delta_{\mathrm{s}} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ captures sensor attacks. In particular, if $\delta_{\mathrm{s}}(\cdot, \cdot)$ is nonzero, then the uncompromised state vector $x(t)$, $t \geq 0$, is corrupted with a faulty (or malicious) signal $\delta_{\mathrm{s}}(\cdot, \cdot)$. Alternatively, if $\delta_{\mathrm{s}}(t, x) \equiv 0$, then $\tilde{x}(t) = x(t)$, $t \geq 0$, and the uncompromised state vector is available for feedback. Furthermore, we assume that the control input is also compromised and is given by

$$\tilde{u}(t) = u(t) + \delta_{\mathrm{a}}(t, x(t)), \quad t \geq 0, \tag{2.3}$$

where $\tilde{u}(t) \in \mathbb{R}^m$, $t \geq 0$, and $\delta_{\mathrm{a}} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ captures actuator attacks. In particular, if $\delta_{\mathrm{a}}(\cdot, \cdot)$ is nonzero, then the uncompromised control signal $u(t)$, $t \geq 0$, is corrupted with a faulty (or malicious) signal $\delta_{\mathrm{a}}(\cdot, \cdot)$. Alternatively, if $\delta_{\mathrm{a}}(t, x) \equiv 0$, then $\tilde{u}(t) = u(t)$, $t \geq 0$, and the control signal is uncompromised; see Figure 2.2.1.

Since (A, B) is controllable, there exists a feedback gain matrix $K \in \mathbb{R}^{m \times n}$ such that $A_{\mathrm{r}} \triangleq A + BK$ is Hurwitz. In this case, it follows from converse Lyapunov theory [51] that for every positive definite matrix $R \in \mathbb{R}^{n \times n}$, there exists a unique positive-definite $P \in \mathbb{R}^{n \times n}$

satisfying

$$0 = A_{\mathrm{r}}^{\mathrm{T}}P + PA_{\mathrm{r}} + R. \tag{2.4}$$

For $\delta_{\mathrm{s}}(t, x(t)) \neq 0$, $t \geq 0$, and $\delta_{\mathrm{a}}(t, x(t)) \neq 0$, $t \geq 0$, our objective is to design a controller $\mathcal{G}_{\mathrm{c}}$ of the form

$$u(t) = K\tilde{x}(t) + v(t), \quad t \geq 0, \tag{2.5}$$

where $v(t) \in \mathbb{R}^m$, $t \geq 0$, is a corrective signal that suppresses or counteracts the effect of state-dependent sensor and actuator attacks $\delta_{\mathrm{s}}(t, x(t))$, $t \geq 0$, and $\delta_{\mathrm{a}}(t, x(t))$, $t \geq 0$, to asymptotically (or approximately) recover the ideal system performance achieved when the uncompromised state vector is available for feedback and control signal is uncompromised.

2.3. Adaptive Ultimate Boundedness and Stabilization for State-Dependent Sensor and Actuator Attacks

Here, we design the corrective signal $v(t)$, $t \geq 0$, in (2.5) to achieve adaptive ultimate boundedness and stabilization in the presence of state-dependent sensor and actuator attacks. We assume that the sensor attack in (2.2) is parameterized as $\delta_{\mathrm{s}}(t, x(t)) = w(t)x(t)$, $t \geq 0$, where $w(t) \in \mathbb{R}$, $t \geq 0$, is an *unknown* time-varying weight such that $\|w(t)\|_2 \leq \overline{w}$, $t \geq 0$, and $\|\dot{w}(t)\|_2 \leq \overline{\dot{w}}$, $t \geq 0$, with *unknown* bounds $\overline{w}$ and $\overline{\dot{w}}$. In this case, we assume that $w(t) > -1$, $t \geq 0$, in order to construct a feasible corrective signal $v(t)$, $t \geq 0$, since $w(t) \equiv -1$ results in $\tilde{x}(t) \equiv 0$, and hence, it is not possible to construct $v(t)$, $t \geq 0$, to asymptotically recover the ideal system performance.

Furthermore, we assume that the actuator attack in (2.3) can be parameterized as $\delta_{\mathrm{a}}(t, x(t)) = W^{\mathrm{T}}(t)\varphi(x(t))$, $t \geq 0$, where $W(t) \in \mathbb{R}^{p\times m}$, $t \geq 0$, is an *unknown* time-varying weighting matrix and $\varphi(x(t)) \in \mathbb{R}^p$, $t \geq 0$, is a nonlinear function with a known structure and with $x(t)$, $t \geq 0$, in general being unknown. Since the uncompromised state vector $x(t)$,

$t \geq 0$, is not available for feedback, we rewrite

$$W^{\mathrm{T}}(t)\varphi(x(t)) = W^{\mathrm{T}}(t)\varphi(\tilde{x}(t)) + \sigma(t, x(t)), \quad t \geq 0, \tag{2.6}$$

where $\sigma(t, x(t)) \in \mathbb{R}^m$, $t \geq 0$, is *unknown* and bounded, that is, $\|\sigma(t, x(t))\|_2 \leq \bar{\sigma}$, $t \geq 0$, and where $\bar{\sigma} > 0$ is *unknown*. Note that assuming that $\|\sigma(t, x(t))\|_2 \leq \bar{\sigma}$, $t \geq 0$, is without loss of generality since a worst-case actuator attack will lead to actuator amplitude saturation in practice. Therefore, (2.1) can be equivalently written as

$$\dot{x}(t) = Ax(t) + B\Big[u(t) + W^{\mathrm{T}}(t)\varphi(\tilde{x}(t)) + \sigma(t, x(t))\Big], \quad x(0) = x_0, \quad t \geq 0. \tag{2.7}$$

To achieve system ultimate boundedness in the face of time-varying, state-dependent sensor and actuator attacks, we use the corrective signal given by

$$v(t) = -\hat{\mu}(t)K\tilde{x}(t) - \hat{W}^{\mathrm{T}}(t)\varphi(\tilde{x}(t) - \hat{\sigma}(t)\mathrm{sgn}_{\mathrm{v}}(B^{\mathrm{T}}P\tilde{x}(t)), \quad t \geq 0, \tag{2.8}$$

where, for $y \in \mathbb{R}^n$, $\mathrm{sgn}_{\mathrm{v}}(y) \triangleq [\mathrm{sgn}(y_1), \ldots, \mathrm{sgn}(y_n)]^{\mathrm{T}}$, $\mathrm{sgn}(\alpha) \triangleq \frac{\alpha}{|\alpha|}$, $\alpha \neq 0$, and $\mathrm{sgn}(0) \triangleq 0$, and

$$\dot{\hat{\mu}}(t) = \gamma \mathrm{Proj}\big[\hat{\mu}(t),\ \tilde{x}^{\mathrm{T}}(t)PBK\tilde{x}(t)\big], \quad \hat{\mu}(0) = \hat{\mu}_0, \quad t \geq 0, \tag{2.9}$$

$$\dot{\hat{W}}(t) = \eta \mathrm{Proj}_{\mathrm{m}}\big[\hat{W}(t),\ \varphi(\tilde{x}(t))\tilde{x}^{\mathrm{T}}(t)PB\big], \quad \hat{W}(0) = \hat{W}_0, \tag{2.10}$$

$$\dot{\hat{\sigma}}(t) = \nu \mathrm{Proj}\big[\hat{\sigma}(t),\ \|\tilde{x}^{\mathrm{T}}(t)PB\|_1\big], \quad \hat{\sigma}(0) = \hat{\sigma}_0, \tag{2.11}$$

where $\hat{\mu}(t) \in \mathbb{R}$, $t \geq 0$, is the estimate of $\mu(t) \triangleq w(t)(1 + w(t))^{-1} \in \mathbb{R}$, $t \geq 0$, that depends on the sensor uncertainty $w(t)$, $t \geq 0$, $\hat{W}(t) \in \mathbb{R}^{p \times m}$, $t \geq 0$, is the estimate of the parametric uncertainty $W(t)$, $t \geq 0$, $\hat{\sigma}(t) \in \mathbb{R}$, $t \geq 0$, is the estimate of the unknown bound $\bar{\sigma}$, $\gamma \in \mathbb{R}$, $\eta \in \mathbb{R}$, and $\nu \in \mathbb{R}$ are positive design gains, and $\mathrm{Proj} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection operator. Specifically, for a continuously differentiable convex function $\phi : \mathbb{R}^n \to \mathbb{R}$ given by $\phi(\theta) \triangleq \frac{(\varepsilon_\theta + 1)\theta^{\mathrm{T}}\theta - \theta_{\max}^2}{\varepsilon_\theta \theta_{\max}^2}$, where $\theta_{\max} \in \mathbb{R}$ is a *projection norm bound* imposed on $\theta \in \mathbb{R}^n$ and $\varepsilon_\theta > 0$ is a *projection tolerance bound*, the *projection operator* $\mathrm{Proj} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\mathrm{Proj}(\theta, y) \triangleq \begin{cases} y, & \text{if } \phi(\theta) < 0, \\ y, & \text{if } \phi(\theta) \geq 0 \text{ and } \phi'(\theta) y \leq 0 \ , \\ y - \frac{\phi'^{\mathrm{T}}(\theta)\phi'(\theta) y}{\phi'(\theta)\phi'^{\mathrm{T}}(\theta)} \phi(\theta), & \text{if } \phi(\theta) \geq 0 \text{ and } \phi'(\theta) y > 0, \end{cases} \tag{2.12}$$

where $y \in \mathbb{R}^n$.

Note that it follows from the definition of the projection operator that $(\theta-\theta^*)^{\mathrm{T}}(\mathrm{Proj}(\theta, y)-y) \leq 0$, $\theta^* \in \mathbb{R}^n$ [110]. Furthermore, $\mathrm{Proj_m} : \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m} \to \mathbb{R}^{n\times m}$ defines a generalization of the projection operator to matrices wherein

$$\mathrm{Proj_m}(\Theta, Y) \ = \ (\mathrm{Proj}(\mathrm{col}_1(\Theta), \mathrm{col}_1(Y))), \ldots, \mathrm{Proj}(\mathrm{col}_m(\Theta), \mathrm{col}_m(Y)))), \tag{2.13}$$

where $\Theta \in \mathbb{R}^{n\times m}$, $Y \in \mathbb{R}^{n\times m}$, and $\mathrm{col}_i(\cdot)$ denotes the ith column operator. In this case, for a given $\Theta^* \in \mathbb{R}^{n\times m}$, it follows from (2.12) that

$$\mathrm{tr}[(\Theta - \Theta^*)^{\mathrm{T}}(\mathrm{Proj_m}(\Theta - Y) - Y)] = \sum_{i=1}^{m} [\mathrm{col}_i(\Theta - \Theta^*)^{\mathrm{T}}(\mathrm{Proj}(\mathrm{col}_i(\Theta), \mathrm{col}_i(Y)) - \mathrm{col}_i(Y))] \leq 0 \tag{2.14}$$

holds. In this chapter, we assume that the projection norm bound imposed on each column of $\Theta \in \mathbb{R}^{n\times m}$ is $\theta_{\max}$.

Next, define $\mu_\lambda(t) \triangleq \tilde{\mu}(t)\lambda^{\frac{1}{2}}(t)$, $t \geq 0$, $W_\lambda(t) \triangleq \tilde{W}(t)\lambda^{\frac{1}{2}}(t)$, $t \geq 0$, and $\sigma_\lambda(t) \triangleq \tilde{\sigma}(t)\lambda^{\frac{1}{2}}(t)$, $t \geq 0$, with $\tilde{\mu}(t) \triangleq \mu(t) - \hat{\mu}(t)$, $t \geq 0$, $\tilde{W}(t) \triangleq W(t) - \hat{W}(t)$, $t \geq 0$, $\tilde{\sigma}(t) \triangleq \bar{\sigma} - \hat{\sigma}(t)$, $t \geq 0$, and $\lambda(t) \triangleq \big(1 + w(t)\big)^{-1}$, $t \geq 0$. Since $w(t) > -1$, note that $\mu(t)$, $t \geq 0$, and $\lambda(t)$, $t \geq 0$, are well-defined and $\lambda(t) > 0$, $t \geq 0$. For the statement of the next result, note that

$$\begin{aligned} \dot{x}(t) \ &= \ A_{\mathrm{r}}x(t) + \mu_\lambda(t)\lambda^{-\frac{1}{2}}(t)BK\tilde{x}(t) + BW_\lambda^{\mathrm{T}}(t)\lambda^{-\frac{1}{2}}(t)\varphi(\tilde{x}(t)) \\ &\quad + B(\sigma(t, x(t)) - \hat{\sigma}(t)\mathrm{sgn_v}(B^{\mathrm{T}}P\tilde{x}(t))), \quad x(0) = x_0, \quad t \geq 0, \end{aligned} \tag{2.15}$$

$$\dot{\mu}_\lambda(t) \ = \ [\dot{\mu}(t) - \gamma \mathrm{Proj}\big(\hat{\mu}(t),\ \tilde{x}^{\mathrm{T}}(t)PBK\tilde{x}(t)\big)]\lambda^{\frac{1}{2}}(t) + \frac{1}{2}\mu_\lambda(t)\dot{\lambda}(t)\lambda^{-1}(t), \quad \mu_\lambda(0) = \mu_{\lambda 0}, \tag{2.16}$$

$$\begin{aligned} \dot{W}_\lambda(t) \ &= \ [\dot{W}(t) - \eta \mathrm{Proj_m}\big(\hat{W}(t), \varphi(\tilde{x}(t))\tilde{x}^{\mathrm{T}}(t)PB\big)]\lambda^{\frac{1}{2}}(t) \\ &\quad + \frac{1}{2}W_\lambda(t)\dot{\lambda}(t)\lambda^{-1}(t), \quad W_\lambda(0) = W_{\lambda 0}, \end{aligned} \tag{2.17}$$

$$\dot{\sigma}_\lambda(t) \ = \ [-\nu \mathrm{Proj}\big(\hat{\sigma}(t),\ \|\tilde{x}^{\mathrm{T}}(t)BP\|_1\big)]\lambda^{\frac{1}{2}}(t) + \frac{1}{2}\sigma_\lambda(t)\dot{\lambda}(t)\lambda^{-1}(t), \quad \sigma_\lambda(0) = \sigma_{\lambda 0}. \tag{2.18}$$

Theorem 2.1. Consider the linear dynamical system $\mathcal{G}$ given by (2.1) with time-varying, state-dependent sensor and actuator attacks given by (2.2) and (2.3), respectively, where $\|w(t)\|_2 \le \overline{w}$, $t \ge 0$, $\|\dot{w}(t)\|_2 \le \overline{\dot{w}}$, $t \ge 0$, $\|W(t)\|_{\mathrm{F}} \le \overline{W}$, $t \ge 0$, $\|\dot{W}(t)\|_{\mathrm{F}} \le \overline{\dot{W}}$, $t \ge 0$, and $\|\sigma(t, x(t))\|_2 \le \overline{\sigma}$, $t \ge 0$. Then, with the controller $\mathcal{G}_{\mathrm{c}}$ given by (2.5) and the corrective signal $v(t)$, $t \ge 0$, given by (2.8), the closed-loop system given by (2.15)–(2.18) is uniformly bounded for all $\big(x_0, \mu_{\lambda 0}, W_{\lambda 0}, \sigma_{\lambda 0}\big) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{p \times m} \times \mathbb{R}$ with the ultimate bounds

$$\begin{aligned}
\|x(t)\|_2 \le{}& \Bigg[\frac{1}{\lambda_{\min}(P)}[\lambda_{\max}(P) d_1^{-1} d_2 + \gamma^{-1}\overline{\lambda}\big(\overline{\mu} + \hat{\mu}_{\max}\big)^2 + \eta^{-1}\overline{\lambda}\big(\overline{W} + \hat{W}_{\max}\big)^2 \\
&+ \nu^{-1}\overline{\lambda}\big(\overline{\sigma} + \hat{\sigma}_{\max}\big)^2]\Bigg]^{\frac{1}{2}}, \quad t \ge T, && (2.19)\\
|\mu_\lambda(t)| \le{}& [\gamma\lambda_{\max}(P) d_1^{-1} d_2 + \overline{\lambda}\big(\overline{\mu} + \hat{\mu}_{\max}\big)^2 + \gamma\eta^{-1}\overline{\lambda}\big(\overline{W} + \hat{W}_{\max}\big)^2 \\
&+ \gamma\nu^{-1}\overline{\lambda}\big(\overline{\sigma} + \hat{\sigma}_{\max}\big)^2]^{\frac{1}{2}}, \quad t \ge T, && (2.20)\\
\|W_\lambda(t)\|_{\mathrm{F}} \le{}& [\eta\lambda_{\max}(P) d_1^{-1} d_2 + \eta\gamma^{-1}\overline{\lambda}\big(\overline{\mu} + \hat{\mu}_{\max}\big)^2 + \overline{\lambda}\big(\overline{W} + \hat{W}_{\max}\big)^2 \\
&+ \eta\nu^{-1}\overline{\lambda}\big(\overline{\sigma} + \hat{\sigma}_{\max}\big)^2]^{\frac{1}{2}}, \quad t \ge T, && (2.21)\\
|\sigma_\lambda(t)| \le{}& [\nu\lambda_{\max}(P) d_1^{-1} d_2 + \nu\gamma^{-1}\overline{\lambda}\big(\overline{\mu} + \hat{\mu}_{\max}\big)^2 \\
&+ \nu\eta^{-1}\overline{\lambda}\big(\overline{W} + \hat{W}_{\max}\big)^2 + \overline{\lambda}\big(\overline{\sigma} + \hat{\sigma}_{\max}\big)^2]^{\frac{1}{2}}, \quad t \ge T, && (2.22)
\end{aligned}$$

where $T > 0$, $d_1 \triangleq \lambda_{\min}(R)$, $d_2 \triangleq \gamma^{-1}[2\big(\overline{\mu} + \hat{\mu}_{\max}\big)\overline{\dot{\mu}}\overline{\lambda} + \big(\overline{\mu} + \hat{\mu}_{\max}\big)^2\overline{\dot{\lambda}}] + \eta^{-1}[2\big(\overline{W} + \hat{W}_{\max}\big)\overline{\dot{W}}\overline{\lambda} + \big(\overline{W} + \hat{W}_{\max}\big)^2\overline{\dot{\lambda}}] + \nu^{-1}[\big(\overline{\sigma} + \hat{\sigma}_{\max}\big)^2\overline{\dot{\lambda}}]$, and $\hat{\mu}_{\max} \in \mathbb{R}$, $\hat{W}_{\max} \in \mathbb{R}$, and $\hat{\sigma}_{\max} \in \mathbb{R}$ are projection norm bounds.

Proof. To show boundedness of the closed-loop system given by (2.15)–(2.18), consider the Lyapunov-like function given by

$$V\big(x, \mu_\lambda, W_\lambda, \sigma_\lambda\big) = x^{\mathrm{T}} P x + \gamma^{-1}\mu_\lambda^2 + \eta^{-1}\mathrm{tr}(W_\lambda^{\mathrm{T}} W_\lambda) + \nu^{-1}\sigma_\lambda^2, \tag{2.23}$$

where P satisfies (2.4). Note that $V(0,0,0,0) = 0$, $V\big(x, \mu_\lambda, W_\lambda, \sigma_\lambda\big) > 0$, $\big(x, \mu_\lambda, W_\lambda, \sigma_\lambda\big) \ne (0,0,0,0)$, and $V\big(x, \mu_\lambda, W_\lambda, \sigma_\lambda\big)$ is radially unbounded. The time derivative of (2.23) along the closed-loop system trajectories of (2.15)–(2.18) is given by

$$\dot{V}\big(x(t), \mu_\lambda(t), W_\lambda(t), \sigma_\lambda(t)\big) = -x^{\mathrm{T}}(t) R x(t) + 2\mu_\lambda(t)\lambda^{-\frac{1}{2}}(t) x^{\mathrm{T}}(t) P B K \tilde{x}(t)$$

$$
\begin{aligned}
&+2\gamma^{-1}\mu_\lambda(t)[\dot{\mu}(t)-\gamma\mathrm{Proj}\big(\hat{\mu}(t),\tilde{x}^{\mathrm{T}}(t)PBK\tilde{x}(t)\big)]\lambda^{\frac{1}{2}}(t)\\
&+\gamma^{-1}\mu_\lambda^2(t)\dot{\lambda}(t)\lambda^{-1}(t)+2x^{\mathrm{T}}(t)PB\phi_\lambda(t)\lambda^{-\frac{1}{2}}(t)\\
&+2\mathrm{tr}[W_\lambda^{\mathrm{T}}(t)\lambda^{-\frac{1}{2}}(t)\varphi(\tilde{x}(t))x^{\mathrm{T}}(t)PB]\\
&+2\eta^{-1}\mathrm{tr}[W_\lambda^{\mathrm{T}}(t)[\dot{W}(t)-\eta\mathrm{Proj_m}\big(\hat{W}(t),\ \varphi(\tilde{x}(t))\tilde{x}^{\mathrm{T}}(t)PB\big)]\lambda^{\frac{1}{2}}(t)]\\
&+\eta^{-1}\mathrm{tr}(W_\lambda^{\mathrm{T}}(t)W_\lambda(t))\dot{\lambda}(t)\lambda^{-1}(t)\\
&+2x^{\mathrm{T}}(t)PB(\sigma(t,x(t))-\hat{\sigma}(t)\mathrm{sgn_v}(B^{\mathrm{T}}P\tilde{x}(t)))\\
&+2\nu^{-1}\sigma_\lambda(t)[-\nu\mathrm{Proj}\big(\hat{\sigma}(t),\ \|\tilde{x}^{\mathrm{T}}(t)BP\|_1\big)]\lambda^{\frac{1}{2}}(t)\\
&+\nu^{-1}\sigma_\lambda^2(t)\dot{\lambda}(t)\lambda^{-1}(t).
\end{aligned}
\tag{2.24}
$$

Now, using

$$
\begin{aligned}
&2x^{\mathrm{T}}(t)PB(\sigma(t,x(t))-\hat{\sigma}(t)\mathrm{sgn_v}(B^{\mathrm{T}}P\tilde{x}(t)))\\
&\qquad=\ 2\tilde{x}^{\mathrm{T}}(t)PB\sigma(t,x(t))\lambda(t)-2\tilde{x}^{\mathrm{T}}(t)PB\hat{\sigma}(t)\lambda(t)\mathrm{sgn_v}(B^{\mathrm{T}}P\tilde{x}(t))\\
&\qquad\le\ 2\|\tilde{x}^{\mathrm{T}}(t)PB\|_1\overline{\sigma}\lambda(t)-2\|\tilde{x}^{\mathrm{T}}(t)PB\|_1\hat{\sigma}(t)\lambda(t)\\
&\qquad=\ 2\|\tilde{x}^{\mathrm{T}}(t)PB\|_1\sigma_\lambda(t)\lambda^{\frac{1}{2}}(t),\quad t\ge 0,
\end{aligned}
\tag{2.25}
$$

it follows from (2.24) that

$$
\begin{aligned}
&\dot{V}\big(x(t),\mu_\lambda(t),W_\lambda(t),\sigma_\lambda(t)\big)\\
&\quad\le\ -x^{\mathrm{T}}(t)Rx(t)+2\mu_\lambda(t)\lambda^{\frac{1}{2}}(t)\tilde{x}^{\mathrm{T}}(t)PBK\tilde{x}(t)+\gamma^{-1}\mu_\lambda^2(t)\dot{\lambda}(t)\lambda^{-1}(t)+2\gamma^{-1}\mu_\lambda(t)[\dot{\mu}(t)\\
&\qquad\quad-\gamma\mathrm{Proj}\big(\hat{\mu}(t),\ \tilde{x}^{\mathrm{T}}(t)PBK\tilde{x}(t)\big)]\lambda^{\frac{1}{2}}(t)+2\mathrm{tr}[W_\lambda^{\mathrm{T}}(t)\lambda^{\frac{1}{2}}(t)\varphi(\tilde{x}(t))\tilde{x}^{\mathrm{T}}(t)PB]\\
&\qquad+2\eta^{-1}\mathrm{tr}[W_\lambda(t)[\dot{W}(t)-\eta\mathrm{Proj_m}\big(\hat{W}(t),\ \varphi(\tilde{x}(t))\tilde{x}^{\mathrm{T}}(t)PB\big)]\lambda^{\frac{1}{2}}(t)]\\
&\qquad+\eta^{-1}\mathrm{tr}(W_\lambda^{\mathrm{T}}(t)W_\lambda(t))\dot{\lambda}(t)\lambda^{-1}(t)+2\|\tilde{x}^{\mathrm{T}}(t)BP\|_1\sigma_\lambda(t)\lambda^{\frac{1}{2}}(t)\\
&\qquad+2\nu^{-1}\sigma_\lambda(t)[-\nu\mathrm{Proj}\big(\hat{\sigma}(t),\ \|\tilde{x}^{\mathrm{T}}(t)BP\|_1\big)]\lambda^{\frac{1}{2}}(t)+\nu^{-1}\sigma_\lambda^2(t)\dot{\lambda}(t)\lambda^{-1}(t),\quad t\ge 0,
\end{aligned}
\tag{2.26}
$$

where we used the fact that $x(t)=\lambda(t)\tilde{x}(t)$, $t\ge 0$.

Next, for $t\ge 0$, using

$$
\mu_\lambda(t)\lambda^{\frac{1}{2}}(t)\tilde{x}^{\mathrm{T}}(t)PBK\tilde{x}(t)-\mu_\lambda(t)\lambda^{\frac{1}{2}}(t)\mathrm{Proj}\big(\hat{\mu}(t),\ \tilde{x}^{\mathrm{T}}(t)PBK\tilde{x}(t)\big)
$$

$$= \lambda(t)\big(\hat{\mu}(t) - \mu(t)\big)[\mathrm{Proj}\big(\hat{\mu}(t), \tilde{x}^{\mathrm{T}}(t)PBK\tilde{x}(t)\big) - \tilde{x}^{\mathrm{T}}(t)PBK\tilde{x}(t)] \leq 0, \quad (2.27)$$

$$\mathrm{tr}[W_\lambda(t)\lambda^{\frac{1}{2}}(t)\varphi(\tilde{x}(t))\tilde{x}^{\mathrm{T}}(t)PB] - \eta^{-1}\mathrm{tr}[W_\lambda(t)[\eta\mathrm{Proj}_{\mathrm{m}}[\hat{W}(t),\ \varphi(\tilde{x}(t))\tilde{x}^{\mathrm{T}}(t)PB]]\lambda^{\frac{1}{2}}(t)] \leq 0, \quad (2.28)$$

$$2\|\tilde{x}^{\mathrm{T}}(t)BP\|_1\sigma_\lambda(t)\lambda^{\frac{1}{2}}(t) + 2\nu^{-1}\sigma_\lambda(t)[-\nu\mathrm{Proj}\big(\hat{\sigma}(t),\ \|\tilde{x}^{\mathrm{T}}(t)BP\|_1\big)]\lambda^{\frac{1}{2}}(t) \leq 0, \quad (2.29)$$

it follows from (2.26) that

$$\begin{aligned} \dot{V}\big(x(t), \mu_\lambda(t), W_\lambda(t), \sigma_\lambda(t)\big) &\leq -x^{\mathrm{T}}(t)Rx(t) + 2\gamma^{-1}\mu_\lambda(t)\dot{\mu}(t) + \gamma^{-1}\mu_\lambda^2(t)\dot{\lambda}(t)\lambda^{-1}(t) \\ &\quad +2\eta^{-1}\mathrm{tr}[W_\lambda^{\mathrm{T}}(t)\dot{W}_\lambda(t)] + \eta^{-1}\mathrm{tr}[W_\lambda^{\mathrm{T}}(t)W_\lambda(t)\dot{\lambda}(t)\lambda^{-1}(t)] \\ &\quad +\nu^{-1}\sigma_\lambda^2(t)\dot{\lambda}(t)\lambda^{-1}(t) \\ &\leq -d_1\|x(t)\|_2^2 + d_2, \quad t \geq 0, \end{aligned} \quad (2.30)$$

and hence, $\dot{V}\big(x(t), \mu_\lambda(t), W_\lambda(t), \sigma_\lambda(t)\big) < 0$ outside of the compact set

$$\begin{aligned} \mathcal{D}_{\mathrm{c}} \triangleq \Big\{&\big(x, \mu_\lambda, W_\lambda, \sigma_\lambda\big) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{p\times m} \times \mathbb{R} : \|x\|_2 \leq \vartheta_1, \quad |\mu_\lambda| \leq \vartheta_2, \\ &\|W_\lambda\|_{\mathrm{F}} \leq \vartheta_3, \quad \text{and} \quad |\sigma_\lambda| \leq \vartheta_4\Big\}, \end{aligned} \quad (2.31)$$

where $\vartheta_1 \triangleq \sqrt{d_2/d_1}$, $\vartheta_2 \triangleq \overline{\lambda}^{\frac{1}{2}}\big(\overline{\mu} + \hat{\mu}_{\max}\big)$, $\vartheta_3 \triangleq \overline{\lambda}^{\frac{1}{2}}\big(\overline{W} + \hat{W}_{\max}\big)$, and $\vartheta_4 \triangleq \overline{\lambda}^{\frac{1}{2}}\big(\overline{\sigma} + \hat{\sigma}_{\max}\big)$. This proves uniform boundedness of the solution $\big(x(t), \mu_\lambda(t), W_\lambda(t), \sigma_\lambda(t)\big)$ of the closed-loop system given by (2.15)–(2.18) for all $\big(x_0, \mu_{\lambda 0}, W_{\lambda 0}, \sigma_{\lambda 0}\big) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{p\times m} \times \mathbb{R}$ [51].

To show the ultimate bounds for $x(t)$, $t \geq T$, $\mu_\lambda(t)$, $t \geq T$, $W_\lambda(t)$, $t \geq T$, and $\sigma_\lambda(t)$, $t \geq T$, given by (2.19)–(2.22), note that, for $t \geq T$,

$$\lambda_{\min}(P)\|x(t)\|_2^2 + \gamma^{-1}|\mu_\lambda(t)|^2 + \eta^{-1}\|W_\lambda(t)\|_{\mathrm{F}}^2 + \nu^{-1}|\sigma_\lambda(t)|^2 \leq \upsilon_{\max}, \quad (2.32)$$

where $\upsilon_{\max} \triangleq \lambda_{\min}(P)\vartheta_1^2 + \gamma^{-1}\vartheta_2^2 + \eta^{-1}\vartheta_3^2 + \nu^{-1}\vartheta_4^2$. It now follows from (2.32) that $\|x(t)\|_2^2 \leq \frac{\upsilon_{\max}}{\lambda_{\min}(P)}$, $t \geq T$, $|\mu_\lambda(t)|^2 \leq \gamma\upsilon_{\max}$, $t \geq T$, $\|W_\lambda(t)\|_{\mathrm{F}}^2 \leq \eta\upsilon_{\max}$, $t \geq T$, and $|\sigma_\lambda(t)|^2 \leq \nu\upsilon_{\max}$, $t \geq T$. The result is now immediate. $\square$

Theorem 2.1 assumes that $w(t) > -1$, $t \geq 0$. This assumption implies that $\lambda(t) > 0$, $t \geq 0$. As long as the sign of $\lambda(t)$ is known, Theorem 2.1 can be used to address the case where $\lambda(t) < 0$, $t \geq 0$. The assumption $w(t) > -1$, $t \geq 0$, can be relaxed by utilizing tools from [73] that can allow $\lambda(t)$ to have any sign as long as $w(t) \neq -1$ under the assumption that its sign is a priori known.

Note that the controller $u(t)$, $t \geq 0$, given by (2.5) is discontinuous because of the presence of the signum function $\mathrm{sgn_v}(\cdot)$ in the controller architecture. This discontinuity can lead to a chattering phenomenon, which is undesirable in practice. In order to reduce or eliminate the chattering effect, a smooth function can be implemented instead of the signum function [109]; that is, we replace $\mathrm{sgn_v}(\cdot)$ by $\mathrm{tanh_v}(\cdot)$, where, for $y \in \mathbb{R}^n$, $\mathrm{tanh_v}(y) \triangleq [\tanh(y_1), \ldots, \tanh(y_n)]^{\mathrm{T}}$. Note that ([109])

$$0 \leq |\alpha| - \alpha \tanh\left(\frac{\alpha}{\varepsilon}\right) \leq c_0\varepsilon, \qquad \alpha \in \mathbb{R}, \tag{2.33}$$

where $\varepsilon > 0$ is a design constant and c_0 satisfies $c_0 = e^{-(c_0+1)}$, and hence, $c_0 = 0.2785$. Thus, we modify (2.8) as

$$v(t) = -\hat{\mu}(t)K\tilde{x}(t) - \hat{W}^{\mathrm{T}}(t)\varphi(\tilde{x}(t)) - \hat{\sigma}(t)\mathrm{tanh_v}\left(\frac{B^{\mathrm{T}}P\tilde{x}(t)\hat{\sigma}(t)}{\varepsilon}\right), \quad t \geq 0. \tag{2.34}$$

In this case, (2.25) becomes

$$\begin{aligned}
2x^{\mathrm{T}}(t)PB&\left[\sigma(t,x(t)) - \hat{\sigma}(t)\mathrm{tanh_v}\left(\frac{B^{\mathrm{T}}P\tilde{x}(t)\hat{\sigma}(t)}{\varepsilon}\right)\right] \\
&\leq 2\|\tilde{x}^{\mathrm{T}}(t)PB\|_1\overline{\sigma}\lambda(t) - 2\tilde{x}^{\mathrm{T}}(t)PB\hat{\sigma}(t)\lambda(t)\mathrm{tanh_v}\left(\frac{B^{\mathrm{T}}P\tilde{x}(t)\hat{\sigma}(t)}{\varepsilon}\right) \\
&= 2\|\tilde{x}^{\mathrm{T}}(t)PB\|_1\tilde{\sigma}(t)\lambda(t) + \sum_{i=1}^{m} 2\lambda(t)\Bigg[|(\tilde{x}^{\mathrm{T}}(t)PB)_i\hat{\sigma}(t)| \\
&\quad -(\tilde{x}^{\mathrm{T}}(t)PB)_i\hat{\sigma}(t)\mathrm{tanh_v}\left(\frac{(\tilde{x}^{\mathrm{T}}(t)PB)_i\hat{\sigma}(t)}{\varepsilon}\right)\Bigg] \\
&\leq 2\|\tilde{x}^{\mathrm{T}}(t)PB\|_1\sigma_\lambda(t)\lambda^{\frac{1}{2}}(t) + 2m\overline{\lambda}c_0\varepsilon, \quad t \geq 0.
\end{aligned} \tag{2.35}$$

Now, it follows from (2.30) that

$\dot{V}\big(x(t), \mu_\lambda(t), W_\lambda(t), \sigma_\lambda(t)\big)$

$$\leq -x^{\mathrm{T}}(t)Rx(t) + 2\gamma^{-1}\mu_\lambda(t)\dot{\mu}(t) + \gamma^{-1}\mu_\lambda^2(t)\dot{\lambda}(t)\lambda^{-1}(t) + 2\eta^{-1}\mathrm{tr}[W_\lambda^{\mathrm{T}}(t)\dot{W}_\lambda(t)]$$
$$+\eta^{-1}\mathrm{tr}[W_\lambda^{\mathrm{T}}(t)W_\lambda(t)\dot{\lambda}(t)\lambda^{-1}(t)] + \nu^{-1}\sigma_\lambda^2(t)\dot{\lambda}(t)\lambda^{-1}(t) + 2m\overline{\lambda}c_0\varepsilon$$
$$\leq -d_1\|x(t)\|_2^2 + d_3, \quad t \geq 0, \tag{2.36}$$

where $d_3 \triangleq \gamma^{-1}[2(\overline{\mu}+\hat{\mu}_{\max})\overline{\dot{\mu}}\overline{\lambda} + (\overline{\mu}+\hat{\mu}_{\max})^2\overline{\dot{\lambda}}] + \eta^{-1}[2(\overline{W}+\hat{W}_{\max})\overline{\dot{W}}\overline{\lambda} + (\overline{W}+\hat{W}_{\max})^2\overline{\dot{\lambda}}] + \nu^{-1}[(\overline{\sigma}+\hat{\sigma}_{\max})^2\overline{\dot{\lambda}}] + 2m\overline{\lambda}c_0\varepsilon$. Hence, $\dot{V}(x(t),\mu_\lambda(t),W_\lambda(t),\sigma_\lambda(t)) < 0$ outside of the compact set

$$\Omega_{\mathrm{c}} \triangleq \Big\{(x,\mu_\lambda,W_\lambda,\sigma_\lambda) \in \mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^{p\times m}\times\mathbb{R}: \quad \|x\|_2 \leq \vartheta_1,$$
$$|\mu_\lambda| \leq \vartheta_2, \quad \|W_\lambda\|_{\mathrm{F}} \leq \vartheta_3, \quad \text{and} \quad |\sigma_\lambda| \leq \vartheta_4\Big\}, \tag{2.37}$$

where $\vartheta_1 \triangleq \sqrt{d_3/d_1}$, $\vartheta_2 \triangleq \overline{\lambda}^{\frac{1}{2}}(\overline{\mu}+\hat{\mu}_{\max})$, $\vartheta_3 \triangleq \overline{\lambda}^{\frac{1}{2}}(\overline{W}+\hat{W}_{\max})$, and $\vartheta_4 \triangleq \overline{\lambda}^{\frac{1}{2}}(\overline{\sigma}+\hat{\sigma}_{\max})$. This proves uniform boundedness of the solution $(x(t),\mu_\lambda(t),W_\lambda(t),\sigma_\lambda(t))$ of the closed-loop system given by (2.15)–(2.18) for all $(x_0,\mu_{\lambda 0},W_{\lambda 0},\sigma_{\lambda 0}) \in \mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^{p\times m}\times\mathbb{R}$. Ultimate boundedness follows using similar arguments as in the proof of Theorem 2.1.

In arriving at Theorem 2.1 we assumed that $\|\sigma(t,x(t))\|_2 \leq \bar{\sigma}$, $t \geq 0$, where $\bar{\sigma} > 0$ is unknown. Alternatively, we can assume that $\sigma(t,x(t))$ satisfies the Lipschitz condition $\|\sigma(t,x(t))\|_2 \leq \bar{\sigma}\|\tilde{x}(t)\|_2$, $t \geq 0$, where $\bar{\sigma} > 0$ is an *unknown* Lipschitz constant. In this case, it can be shown that Theorem 2.1 holds with

$$v(t) = -\hat{\mu}(t)K\tilde{x}(t) - \hat{W}^{\mathrm{T}}(t)\varphi(\tilde{x}(t)) - \hat{\sigma}(t)\|\tilde{x}(t)\|_2\mathrm{sgn}_{\mathrm{v}}(B^{\mathrm{T}}P\tilde{x}(t)), \quad t \geq 0, \tag{2.38}$$

and with (2.11) and (2.18) replaced by

$$\dot{\hat{\sigma}}(t) = \nu\mathrm{Proj}\big[\hat{\sigma}(t),\ \|\tilde{x}(t)\|_2\|\tilde{x}^{\mathrm{T}}(t)PB\|_1\big], \quad \hat{\sigma}(0) = \hat{\sigma}_0, \tag{2.39}$$

$$\dot{\sigma}_\lambda(t) = [-\nu\mathrm{Proj}\big(\hat{\sigma}(t),\ \|\tilde{x}(t)\|_2\|\tilde{x}^{\mathrm{T}}(t)BP\|_1\big)]\lambda^{\frac{1}{2}}(t) + \frac{1}{2}\sigma_\lambda(t)\dot{\lambda}(t)\lambda^{-1}(t), \quad \sigma_\lambda(0) = \sigma_{\lambda 0}. \tag{2.40}$$

Next, we specialize Theorem 2.1 to the case where the sensor and actuator attacks on the system are time-invariant. In particular, we assume that the compromised system state is given by

$$\tilde{x}(t) = x(t) + \delta_{\rm s}(x(t)), \quad t \geq 0, \tag{2.41}$$

and is available for feedback, where $\tilde{x}(t) \in \mathbb{R}^n$, $t \geq 0$, and $\delta_{\rm s} : \mathbb{R}^n \to \mathbb{R}^n$ captures sensor attacks. Furthermore, we assume that the control input is also compromised and is given by

$$\tilde{u}(t) = u(t) + \delta_{\rm a}(x(t)), \quad t \geq 0, \tag{2.42}$$

where $\tilde{u}(t) \in \mathbb{R}^m$, $t \geq 0$, and $\delta_{\rm a} : \mathbb{R}^n \to \mathbb{R}^m$ captures actuator attacks. Moreover, we assume that the sensor attack in (2.41) is parameterized as $\delta_{\rm s}(x(t)) = wx(t)$, $t \geq 0$, where $w \in \mathbb{R}$ is an *unknown* weight such that $\|w\|_2 \leq \overline{w}$ with *unknown* bound $\overline{w}$. In addition, we assume that the actuator attack in (2.42) is parameterized as $\delta_{\rm a}(x(t)) = W^{\rm T}\varphi(x(t))$, $t \geq 0$, where $W \in \mathbb{R}^{p\times m}$ is an *unknown* weighting matrix and $\varphi(x(t)) \in \mathbb{R}^p$, $t \geq 0$, is a nonlinear function with a known structure and with $x(t)$, $t \geq 0$, in general being unknown. In this case, (2.6) becomes

$$W^{\rm T}\varphi(x(t)) = W^{\rm T}\varphi(\tilde{x}(t)) + \sigma(x(t)), \quad t \geq 0, \tag{2.43}$$

where $\sigma(x(t)) \in \mathbb{R}^m$, $t \geq 0$, is *unknown* and bounded, that is, $\|\sigma(x(t))\|_2 \leq \bar{\sigma}$, $t \geq 0$, and where $\bar{\sigma} > 0$ is *unknown*. Therefore, (2.1) can be equivalently written as

$$\dot{x}(t) = Ax(t) + B\Big[u(t) + W^{\rm T}\varphi(\tilde{x}(t)) + \sigma(x(t))\Big], \quad x(0) = x_0, \quad t \geq 0. \tag{2.44}$$

Next, define $\mu_\lambda(t) \triangleq \tilde{\mu}(t)\lambda^{\frac{1}{2}}$, $t \geq 0$, $W_\lambda(t) \triangleq \tilde{W}(t)\lambda^{\frac{1}{2}}$, $t \geq 0$, and $\sigma_\lambda(t) \triangleq \tilde{\sigma}(t)\lambda^{\frac{1}{2}}$, $t \geq 0$, with $\tilde{\mu}(t) \triangleq \mu - \hat{\mu}(t)$, $t \geq 0$, $\tilde{W}(t) \triangleq W - \hat{W}(t)$, $t \geq 0$, $\tilde{\sigma}(t) \triangleq \bar{\sigma} - \hat{\sigma}(t)$, $t \geq 0$, and $\lambda \triangleq \big(1 + w\big)^{-1}$. For the statement of the next result, note that

$$\begin{aligned}\dot{x}(t) &= A_{\rm r}x(t) + \mu_\lambda(t)\lambda^{-\frac{1}{2}}BK\tilde{x}(t) + BW_\lambda(t)\lambda^{-\frac{1}{2}}\varphi(\tilde{x}(t)) \\ &\quad + B(\sigma(x(t)) - \hat{\sigma}(t)\mathrm{sgn}(B^{\rm T}P\tilde{x}(t))), \quad x(0) = x_0, \quad t \geq 0, \end{aligned} \tag{2.45}$$

$$\dot{\mu}_\lambda(t) = -\gamma\mathrm{Proj}\big(\hat{\mu}(t),\ \tilde{x}^{\rm T}(t)PBK\tilde{x}(t)\big)\lambda^{\frac{1}{2}}, \quad \mu_\lambda(0) = \mu_{\lambda 0}, \tag{2.46}$$

$$\dot{W}_\lambda(t) = -\eta\mathrm{Proj_m}\big(\hat{W}(t),\ \varphi(\tilde{x}(t))\tilde{x}^{\rm T}(t)PB\big)\lambda^{\frac{1}{2}}, \quad W_\lambda(0) = W_{\lambda 0}, \tag{2.47}$$

$$\dot{\sigma}_\lambda(t) \;=\; -\nu \mathrm{Proj}\big(\hat{\sigma}(t),\; \|\tilde{x}^{\mathrm{T}}(t)BP\|_1\big)\lambda^{\frac{1}{2}}, \quad \sigma_\lambda(0) = \sigma_{\lambda 0}. \tag{2.48}$$

Theorem 2.2. Consider the linear dynamical system $\mathcal{G}$ given by (2.1) with time-invariant, state-dependent sensor and actuator attacks given by (2.41) and (2.42), respectively, where $\|w\|_2 \le \overline{w}$, $t \ge 0$, $\|W\|_{\mathrm{F}} \le \overline{W}$, $t \ge 0$, and $\|\sigma(x(t))\|_2 \le \overline{\sigma}$, $t \ge 0$. Then, with the controller $\mathcal{G}_{\mathrm{c}}$ given by (2.5) and the corrective signal $v(t)$, $t \ge 0$, given by (2.8), the closed-loop system given by (2.45)–(2.48) is Lyapunov stable for all $\big(x_0, \mu_{\lambda 0}, W_{\lambda 0}, \sigma_{\lambda 0}\big) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{p\times m} \times \mathbb{R}$ and $\lim_{t\to\infty} x(t) = 0$.

Proof. Since the sensor and actuator attacks are time-invariant, it follows that $\overline{\dot{\mu}} = 0$, $\overline{\dot{\lambda}} = 0$, and $\overline{\dot{W}} = 0$, and hence, d_2 in Theorem 2.1 given by $d_2 = \gamma^{-1}[2\big(\overline{\mu} + \hat{\mu}_{\max}\big)\overline{\dot{\mu}}\overline{\lambda} + \big(\overline{\mu} + \hat{\mu}_{\max}\big)^2\overline{\dot{\lambda}}] + \eta^{-1}[2\big(\overline{W} + \hat{W}_{\max}\big)\overline{\dot{W}}\overline{\lambda} + \big(\overline{W} + \hat{W}_{\max}\big)^2\overline{\dot{\lambda}}] + \nu^{-1}[\big(\overline{\sigma} + \hat{\sigma}_{\max}\big)^2\overline{\dot{\lambda}}] = 0$. In this case, with the Lyapunov function candidate $V\big(x, \mu_\lambda, W_\lambda, \sigma_\lambda\big)$ given by (2.23), using similar arguments as in the proof of Theorem 2.1 it follows that

$$\dot{V}\big(x(t), \mu_\lambda(t), W_\lambda(t), \sigma_\lambda(t)\big) \le -x^{\mathrm{T}}(t)Rx(t) \le -d_1\|x(t)\|_2^2, \quad t \ge 0, \tag{2.49}$$

which shows that the closed-loop system given by (2.45)–(2.48) is Lyapunov stable for all $\big(x_0, \mu_{\lambda 0}, W_{\lambda 0}, \sigma_{\lambda 0}\big) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{p\times m} \times \mathbb{R}$.

Finally, with $W_1\big(x, \mu_\lambda, W_\lambda, \sigma_\lambda\big) = W_2\big(x, \mu_\lambda, W_\lambda, \sigma_\lambda\big) = V\big(x, \mu_\lambda, W_\lambda, \sigma_\lambda\big)$ and $W\big(x, \mu_\lambda, W_\lambda, \sigma_\lambda\big) = x^{\mathrm{T}}Rx$, it follows from Theorem 2.5 of [52] that $\big(x(t), \mu_\lambda(t), W_\lambda(t), \sigma_\lambda(t)\big) \to \mathcal{R}$ as $t \to \infty$, where $\mathcal{R} \triangleq \{\big(x, \mu_\lambda, W_\lambda, \sigma_\lambda\big)\colon W\big(x, \mu_\lambda, W_\lambda, \sigma_\lambda\big) = 0\} = \{\big(x, \mu_\lambda, W_\lambda, \sigma_\lambda\big)\colon x = 0\}$. In particular, note that

$$\begin{aligned} \dot{W}\big(x(t), \mu_\lambda(t), W_\lambda(t), \sigma_\lambda(t)\big) \;&=\; 2x^{\mathrm{T}}(t)R\dot{x}(t) \\ &=\; 2x^{\mathrm{T}}(t)R[A_{\mathrm{r}}x(t) + \mu_\lambda(t)\lambda^{-\frac{1}{2}}BK\tilde{x}(t) + B\phi_\lambda(t)\lambda^{-\frac{1}{2}} \\ &\quad\; + BW_\lambda^{\mathrm{T}}(t)\lambda^{-\frac{1}{2}}\varphi(\tilde{x}(t)) + B(\sigma(x(t)) - \hat{\sigma}(t)\mathrm{sgn}(B^{\mathrm{T}}P\tilde{x}(t)))] \end{aligned}$$

is bounded for all $t \ge 0$, and hence, all conditions of Theorem 2.5 of [52] are satisfied proving that $x(t) \to 0$ as $t \to 0$. $\square$

It is important to note that since $\dot{V}\big(x(t), \mu_\lambda(t), W_\lambda(t), \sigma_\lambda(t)\big)$, $t \geq 0$, is not continuously differentiable, a standard proof involving Barbalat's lemma does not hold in proving partial asymptotic stability in Theorem 2.2. Consequently, Theorem 2.2 requires the more general result given by Theorem 2.5 of [52].

2.4. Illustrative Numerical Example

To illustrate the key ideas presented in this chapter, we consider a dynamical system representing the lateral directional dynamics of an aircraft adopted from [73, p. 136] given by

$$\begin{bmatrix} \dot{\beta}(t) \\ \dot{p}(t) \\ \dot{r}(t) \end{bmatrix} = \begin{bmatrix} -0.025 & 0.104 & -0.994 \\ 574.7 & 0 & 0 \\ 16.20 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta(t) \\ p(t) \\ r(t) \end{bmatrix} + \begin{bmatrix} 0.122 & -0.276 \\ -53.61 & 33.25 \\ 195.5 & -529.4 \end{bmatrix} u(t), \quad t \geq 0, \tag{2.50}$$

where $[\beta(0),\ p(0),\ r(0)]^{\mathrm{T}} = [1,\ -2,\ -1]^{\mathrm{T}}$ and with state feedback control gain

$$K = \begin{bmatrix} 2.053 & 0.079 & -0.045 \\ -3.823 & -0.128 & 0.102 \end{bmatrix}, \tag{2.51}$$

where the state vector $x(t) = \big[\beta(t),\ p(t),\ r(t)\big]^{\mathrm{T}}$, $t \geq 0$, contains the sideslip angle in deg, the roll rate in deg/sec, and the yaw rate in deg/sec, respectively, and the control input $u(t) = \big[\delta_{\mathrm{ail}}(t),\ \delta_{\mathrm{rud}}(t)\big]^{\mathrm{T}}$, $t \geq 0$, contains the aileron command in deg and the rudder command in deg, respectively. The nominal performance of this dynamical system is shown in Figure 2.4.1.

To illustrate the results of Theorem 3.1 with (2.8) replaced by (2.34) consider the time-varying, state-dependent sensor and actuator attacks given by (2.2) and (2.3) respectively, with $w(t) = -\big(0.75 + 0.15\sin(2.5t)\big)$, $t \geq 0$, and $\delta_{\mathrm{a}}(t, x(t)) = [1, \quad 1]^{\mathrm{T}} 0.5\cos(2.5t) + [0.1\cos(2t), \quad 0.5\sin(t)]^{\mathrm{T}} 0.2\sin(\beta(t))\cos(p(t))$, $t \geq 0$. The system performance of the controller $\mathcal{G}_{\mathrm{c}}$ given by (2.5) without any corrective action (i.e., $v(t) \equiv 0$) results in an unstable closed-loop system and is shown in Figure 2.4.2.

To design the proposed corrective signal given by (2.34), (2.10)–(2.11), we set $\gamma = 0.8$,

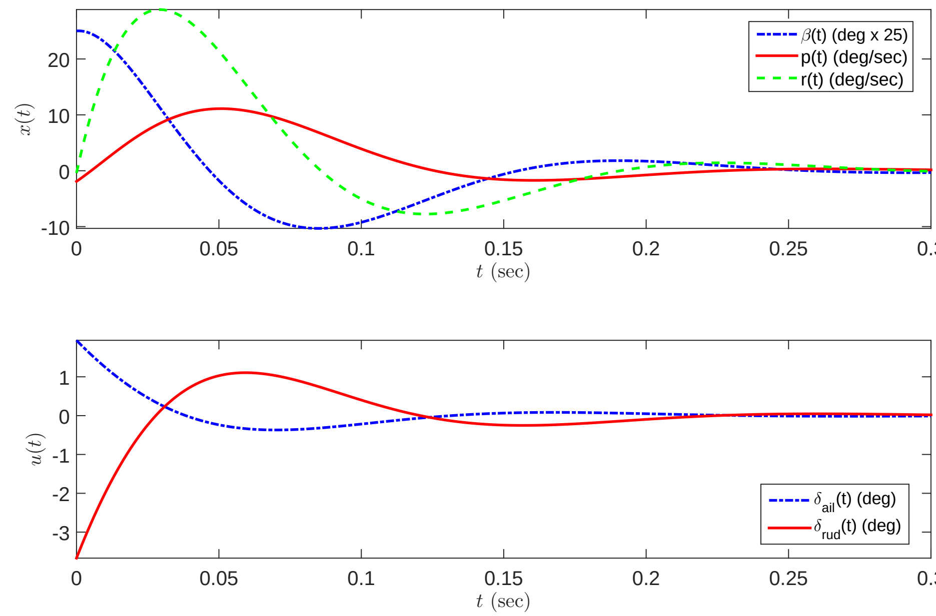


Figure 2.4.1: Nominal system performance of the lateral directional dynamics of the aircraft given by (2.50) when the state vector $x(t)$, $t \geq 0$, is available for feedback and the control signal is uncompromised.

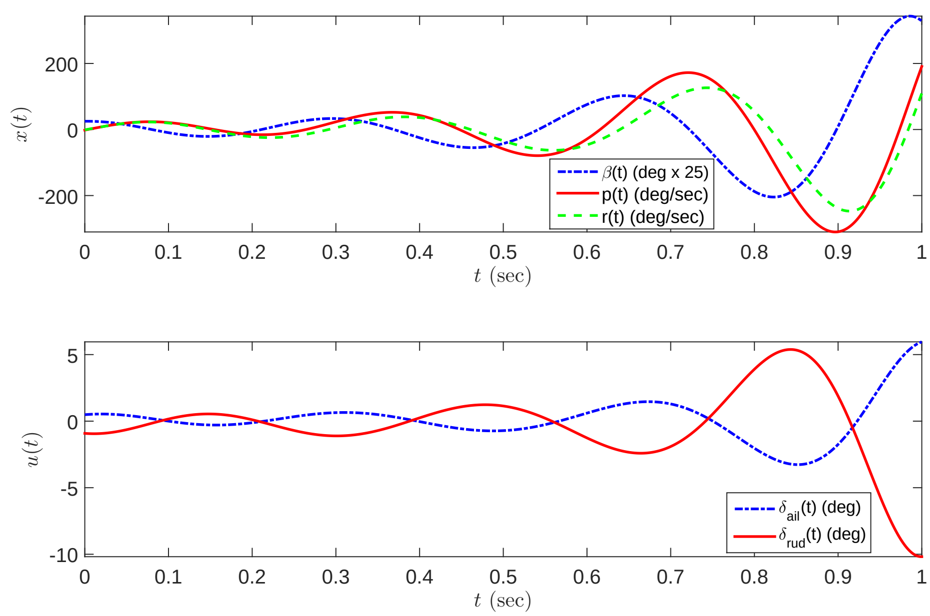


Figure 2.4.2: System performance of the lateral directional dynamics of the aircraft given by (2.50) in the presence of time-varying and state-dependent sensor and actuator attacks without any corrective signal (i.e., $v(t) \equiv 0$) in (2.34).

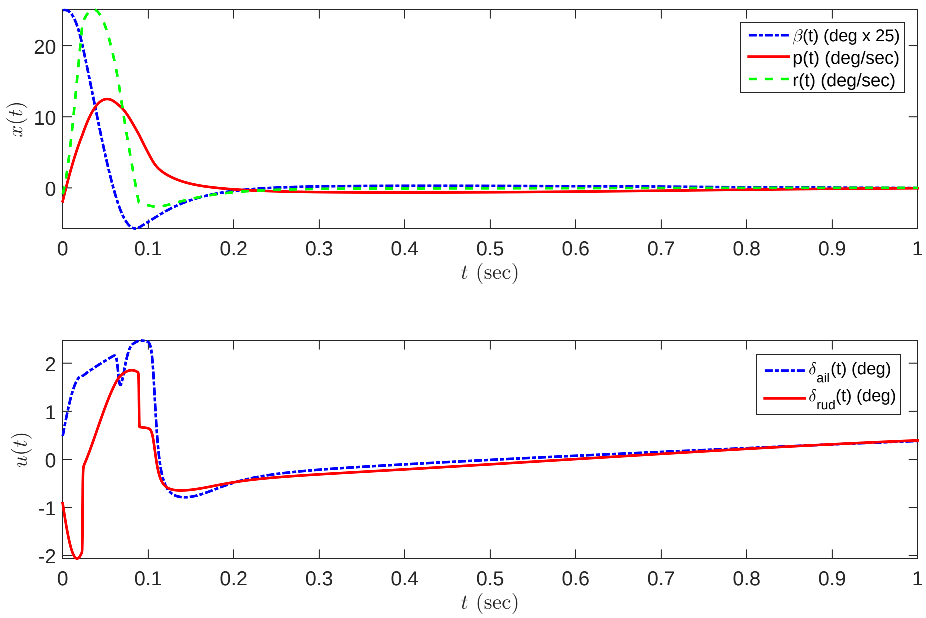


Figure 2.4.3: System performance of the lateral directional dynamics of the aircraft given by (2.50) in the presence of time-varying and state-dependent sensor and actuator attacks with the proposed corrective signal given by (2.34), (2.10)–(2.11) with $\gamma = 0.8$, $\xi = 0.8$, $\eta = 0.8$, $\nu = 0.8$, and $R = I_3$.

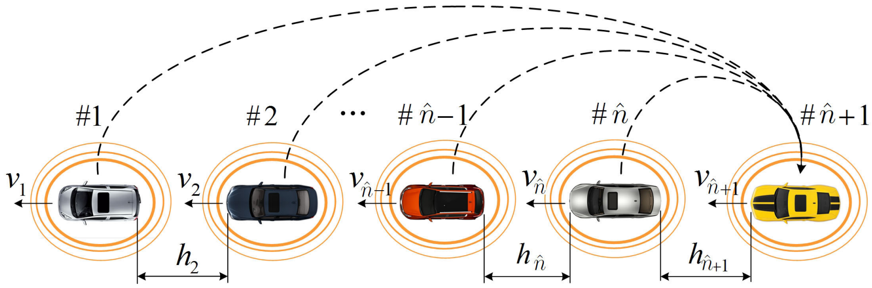


Figure 2.5.1: Platoon formation of the $\hat{n}+1$ vehicles.

$\xi = 0.8$, $\eta = 0.8$, $\nu = 0.8$, and $R = I_3$. Alternatively, a more methodical selection using a convex optimization approach [38] can be used to select the design parameters. The system performance of the controller given by (2.5) with the proposed corrective signal is depicted in Figure 2.4.3. This shows that the proposed adaptive control architecture achieves satisfactory system performance in the face of time-varying, state-dependent sensor and actuator attacks.

2.5. Adaptive Control for Mitigating Sensor and Actuator Attacks in Connected Autonomous Vehicle Platoons

In this section, we extend the results of Sections 2.2–2.4 to develop an adaptive controller for a team of connected vehicles subject to time-invariant, state-dependent sensor and actuator attacks.

Consider a platoon of $\hat{n}+1$ automobile vehicles shown in Figure 2.5.1 traveling rectalinearly, where $h_i(t)$, $t \geq 0$, denotes the bumper-to-bumper distance between vehicle i and its preceding vehicle $i-1$, and $v_i(t)$, $t \geq 0$, denotes the velocity of vehicle i. The $\hat{n}$ forward vehicles, which only transmit position and velocity signals through vehicle-to-vehicle communication, are presumed to be driven by a human. The dynamics of the ith vehicle are given by

$$\dot{h}_i(t) = v_{i-1}(t) - v_i(t), \quad h_i(0) = h_{i0}, \quad t \geq 0, \tag{2.52}$$

$$\dot{v}_i(t) = \alpha_i[f(h_i(t)) - v_i(t)] + \beta_i \dot{h}_i(t), \quad v_i(0) = v_{i0}, \tag{2.53}$$

where $i = 2, 3, \ldots, \hat{n}$, α_i and β_i are human parameters with α_i denoting a headway gain and β_i denoting a relative velocity gain such that $\alpha_i > 0$ and $\alpha_i + \beta_i > 0$, and

$$f(h_i(t)) = \begin{cases} 0, & \text{if} \quad h_i(t) \leq h_{\mathrm{s}}, \\ \frac{1}{2} v_{\mathrm{m}} \Big(1 - \cos\Big(\pi \frac{h_i(t) - h_{\mathrm{s}}}{h_{\mathrm{g}} - h_{\mathrm{s}}}\Big)\Big), & \text{if} \quad h_{\mathrm{s}} < h_i(t) < h_{\mathrm{g}}, \\ v_{\mathrm{m}}, & \text{if} \quad h_{\mathrm{g}} \leq h_i(t). \end{cases} \tag{2.54}$$

Here, $f(\cdot)$ denotes a range policy and implies that vehicle i remains stationary if $h_i \leq h_{\mathrm{s}}$, where h_{s} is the stop headway distance. Moreover, $v_i(t)$, $t \geq 0$, increases as $h_i(t)$, $t \geq 0$, increases over the range $[h_{\mathrm{s}}, h_{\mathrm{g}}]$, where h_{g} is the headway distance for maximum velocity. Additionally, if $h_i \geq h_{\mathrm{g}}$, then vehicle i travels at the maximum velocity v_{m}.

The goal of each driver of the $\hat{n}$ following vehicles is to actuate the vehicle to the desired velocity of the leading vehicle $v_1(t) \equiv v_1$, $t \geq 0$, and to the desired headway $h^* = f^{-1}(v_1)$. Without loss of generality, it is assumed that $0 < v_1 < v_{\mathrm{m}}$. Note that the human parameters α_i and β_i can vary for different drivers.

Defining $\Delta h_i(t) \triangleq h_i(t) - h^*$, $i = 2, \ldots, \hat{n}+1$, and $\Delta v_i(t) \triangleq v_i(t) - v_1$, $i = 2, \ldots, \hat{n}+1$, and linearizing the nonlinear model (2.52) about the equilibrium point (h^*, v_1) we obtain

$$\Delta \dot{h}_i(t) = \Delta v_{i-1}(t) - \Delta v_i(t), \quad \Delta h_i(0) = \Delta h_{i0}, \quad t \geq 0, \tag{2.55}$$

$$\Delta \dot{v}_i(t) = \frac{\alpha_i}{\tau_f} \Delta h_i(t) - (\alpha_i + \beta_i) \Delta v_i(t) + \beta_i \Delta v_{i-1}(t), \quad \Delta v_i(0) = \Delta v_{i0}, \tag{2.56}$$

where $i = 2, \ldots, \hat{n}$ and $\tau_f = 1/f'(h^*)$. Note that $\Delta v_1 = 0$ by definition. The dynamics of the autonomous $(\hat{n}+1)$th vehicle receiving kinematic data from the $\hat{n}$ forward vehicles through vehicle-to-vehicle communication are given by

$$\Delta \dot{h}_{\hat{n}+1}(t) = \Delta v_n(t) - \Delta v_{\hat{n}+1}(t), \quad \Delta h_i(0) = \Delta h_{i0}, \quad t \geq 0, \tag{2.57}$$

$$\Delta \dot{v}_{\hat{n}+1}(t) = u(t), \quad \Delta v_i(0) = \Delta v_{i0}, \tag{2.58}$$

where $u(t)$, $t \geq 0$, is the control input.

Using (2.52)–(2.58), it follows that the dynamics for the connected vehicles are given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \tag{2.59}$$

where $x(t) = [\Delta h_2(t), \Delta v_2(t), \ldots, \Delta h_{\hat{n}+1}(t), \Delta v_{\hat{n}+1}(t)]^{\mathrm{T}}$ and $A \in \mathbb{R}^{2\hat{n}\times 2\hat{n}}$ and $B \in \mathbb{R}^{2\hat{n}}$ are given by

$$A = \begin{bmatrix} F_2 & 0 & \cdots & \cdots & 0 \\ G_3 & F_3 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & G_{\hat{n}} & F_{\hat{n}} & 0 \\ 0 & \cdots & 0 & G_{\hat{n}+1} & F_{\hat{n}+1} \end{bmatrix},$$
$$B = \begin{bmatrix} 0 & \cdots & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}},$$

and where, for $i = 2, \ldots, \hat{n}$ and $j = 3, \ldots, \hat{n}$,

$$F_i = \begin{bmatrix} 0 & -1 \\ \alpha_i/\tau_f & -\alpha_i - \beta_i \end{bmatrix}, \quad G_j = \begin{bmatrix} 0 & 1 \\ 0 & \beta_j \end{bmatrix},$$

and

$$F_{\hat{n}+1} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad G_{\hat{n}+1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The following lemma is needed for the main result of Section 2.5.

Lemma 2.1 [42]. If $\alpha_i > 0$ and $\alpha_i + \beta_i > 0$, $i = 2, \ldots, \hat{n}$, then the pair (A, B) is stabilizable.

Next, we assume that the networked vehicle system given by (2.59) is subject to sensor and actuator attacks so that the compromised system state is given by

$$\tilde{x}(t) = x(t) + \delta_{\mathrm{s}}(x(t)) \tag{2.60}$$

and is available for feedback, where $\tilde{x}(t) \in \mathbb{R}^n$, $t \geq 0$, $n = 2\hat{n}$, and $\delta_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}^n$ captures sensor attacks. Specifically, if $\delta_{\mathrm{s}}(\cdot)$ is nonzero, then the uncompromised state vector $x(t)$, $t \geq 0$, is corrupted by a faulty (or malicious) signal $\delta_{\mathrm{s}}(\cdot)$. Alternatively, if $\delta_{\mathrm{s}}(x(t)) = 0$, then $\tilde{x}(t) = x(t)$, $t \geq 0$, and the uncompromised state vector is available for feedback.

Here, we assume that the sensor attack in (2.60) is parameterized as $\delta_{\mathrm{s}}(x) = qx$, where $q \in \mathbb{R}$ is the sensor uncertainty, and hence, by (2.60) we obtain $\tilde{x} = (1 + q)x$. Thus, we assume that $q > -1$ in order to construct a feasible corrective signal $v(t)$, $t \geq 0$, since $q = -1$ results in $\tilde{x}(t) = 0$, and hence, it is not possible to construct $v(t)$, $t \geq 0$, to asymptotically recover the nominal (i.e., uncompromised) system performance.

Furthermore, we assume that the control input is also compromised and is given by

$$\tilde{u}(t) = u(t) + \delta_{\mathrm{a}}(\tilde{x}(t)), \tag{2.61}$$

where $\tilde{u}(t) \in \mathbb{R}^m$, $t \geq 0$, denotes the compromised control command signal and $\delta_{\mathrm{a}} : \mathbb{R}^n \to \mathbb{R}^m$ captures actuator attacks. In particular, if $\delta_{\mathrm{a}}(\cdot)$ is nonzero, then the uncompromised control signal $u(t)$, $t \geq 0$, is corrupted with a faulty (or malicious) signal $\delta_{\mathrm{a}}(\cdot)$. Alternatively, if $\delta_{\mathrm{a}}(x(t)) = 0$, then $\tilde{u}(t) = u(t)$, $t \geq 0$, and the control signal is uncompromised; see Figure 2.5.2. Note that for generality we have assumed a multi-input control architecture, and hence, for the controller analysis and design framework presented in Section 2.6 we assume $m \geq 1$.

Here, we assume that the actuator attack in (2.61) can be parameterized as

$$\delta_{\mathrm{a}}(\tilde{x}) = W^{\mathrm{T}}\varphi(\tilde{x}) + \sigma(\tilde{x}), \tag{2.62}$$

where $W \in \mathbb{R}^{p \times m}$ is an *unknown* weighting matrix, $\varphi(\cdot) \in \mathbb{R}^p$ is a known nonlinear function, and $\sigma(\tilde{x}) \in \mathbb{R}^m$ is *unknown* and assumed to be bounded, that is, $\|\sigma(\tilde{x})\| \leq \bar{\sigma}$, $\tilde{x} \in \mathbb{R}^n$, and $\bar{\sigma} > 0$ is *unknown*. Note that assuming that $\|\sigma(\tilde{x})\| \leq \bar{\sigma}$, $\tilde{x} \in \mathbb{R}^n$, is without loss of generality since a worst-case actuator attack will lead to actuator amplitude saturation in practice. Alternatively, we can assume that $\sigma(\tilde{x})$, $\tilde{x} \in \mathbb{R}^n$, satisfies a Lipschitz condition.

The sensor attack model captures a multiplicative attack, wherein the attacker can corrupt the sensor measurements in a relative sense. For example, under this multiplicative attack model a malicious attack on a vehicle speed sensor will display a fraction of the vehicle's speed resulting in an unintentional increase in the vehicle's regulated velocity. Alternatively, the actuator attack model is an additive state-dependent signal that accounts for a parameterization of the system attack modes as well as any residual signals.

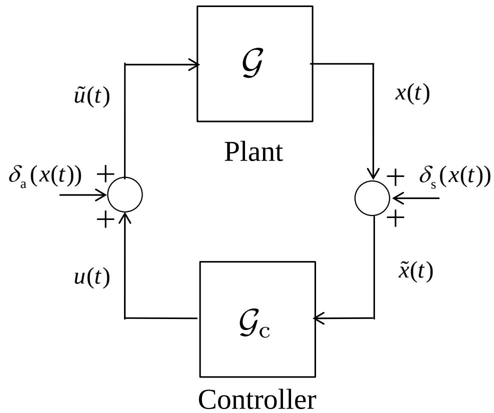


Figure 2.5.2: Closed-loop dynamical system in the presence of sensor and actuator attacks.

Note that the compromised controlled system $\mathcal{G}$ is given by

$$\dot{x}(t) = Ax(t) + B\tilde{u}(t), \quad x(0) = x_0, \quad t \geq 0. \tag{2.63}$$

For $\delta_s(x(t)) \neq 0$, $t \geq 0$, and $\delta_a(x(t)) \neq 0$, $t \geq 0$, our objective is to design a feedback controller $\mathcal{G}_c$ of the form

$$u(t) = K\tilde{x}(t) + v(t), \tag{2.64}$$

where $K \in \mathbb{R}^{m\times n}$ is a feedback gain stabilizing the uncompromised (i.e., nominal) system and $v(t) \in \mathbb{R}^m$, $t \geq 0$, is a corrective signal that suppresses or counteracts the effect of the state-dependent sensor and actuator attacks $\delta_s(x(t))$, $t \geq 0$, and $\delta_a(x(t))$, $t \geq 0$, to approximately recover the nominal system performance achieved when the uncompromised state vector is available for feedback and control signal is uncompromised.

Depending on the available resources of the attacker and the knowledge of the system, different malicious attacks can be injected into cyber-physical systems [137]. These available resources are generally categorized as disclosure and disruption resources. Disclosure resources enable an attacker to gather sensitive information about the system, such as sensor measurements and control input commands, whereas disruption resources enable the attacker to affect and alter the system operation by violating data integrity. Such attacks can include eavesdropping, denial-of-service (DoS), and false data injection.

In our formulation, we are considering data deception (or false data injection) attacks, wherein an attacker modifies the data sent from the sensors to the controller as well as the controller to the actuators. In this type of attacks, the attacker's goal is to prevent the sensor and actuator from receiving correct signals and control packets, and thereby transmitting erroneous information and commands to the sensors and actuators. The sensor attack is modeled as a multiplicative state-dependent sensor attack and the actuator attack is modeled as an additive state-dependent signal to the output of the nominal controller by an attacker. We emphasize that in fault detection and accommodation control, the sensor and actuator faults have a time-varying (but not state-dependent) structure, and hence, fault tolerant control techniques are not usually applicable for dealing with state-dependent sensor and actuator attacks.

It is also important to note that the assumptions and approaches of the fault tolerant schemes considered in the literature (see, for example, [61, 85, 86]) are completely different from the proposed framework. Namely, the main advantages of the proposed approach is that the controller framework is easily implemented for a given closed-loop system without the need of redesigning the nominal controller, and does not need a separate attack detection module.

2.6. Adaptive Control for State-Dependent Sensor and Actuator Attacks

In this section, we design the corrective signal $v(t)$, $t \geq 0$, in (2.64) to achieve ultimate boundedness in the presence of state-dependent sensor and actuator attacks. First, note that since (A, B) is stabilizable there exists a feedback gain matrix $K \in \mathbb{R}^{m \times n}$ such that A_{r} is Hurwitz, where $A_{\mathrm{r}} \triangleq A + BK$. In this case, it follows from converse Lyapunov theory [51] that for every positive-definite matrix $R \in \mathbb{R}^{n \times n}$, there exists a unique positive definite matrix $P \in \mathbb{R}^{n\times n}$ satisfying

$$0 = A_{\mathrm{r}}^{\mathrm{T}} P + P A_{\mathrm{r}} + R. \tag{2.65}$$

To achieve ultimate boundedness in the face of state-dependent sensor and actuator attacks, we use the corrective signal given by

$$v(t) = -\hat{\mu}(t)K\tilde{x}(t) - \hat{W}^{\mathrm{T}}(t)\varphi(\tilde{x}(t)) - \hat{\sigma}(t)\mathrm{sgn}_{\mathrm{v}}(B^{\mathrm{T}}P\tilde{x}(t)), \tag{2.66}$$

where, for $y \in \mathbb{R}^m$, $\mathrm{sgn}_{\mathrm{v}}(y) \triangleq [\mathrm{sgn}(y_1), \ldots, \mathrm{sgn}(y_{\mathrm{m}})]^{\mathrm{T}}$, $\mathrm{sgn}(\alpha) \triangleq \frac{\alpha}{|\alpha|}$, $\alpha \neq 0$, and $\mathrm{sgn}(0) \triangleq 0$, with update laws

$$\dot{\hat{\mu}}(t) = \gamma\tilde{x}^{\mathrm{T}}(t)PBK\tilde{x}(t) - \xi_1\hat{\mu}(t), \quad \hat{\mu}(0) = \hat{\mu}_0, \quad t \geq 0, \tag{2.67}$$

$$\dot{\hat{W}}(t) = \eta\varphi(\tilde{x}(t))\tilde{x}^{\mathrm{T}}(t)PB - \xi_2\hat{W}(t), \quad \hat{W}(0) = \hat{W}_0, \tag{2.68}$$

$$\dot{\hat{\sigma}}(t) = \nu\|\tilde{x}^{\mathrm{T}}(t)PB\|_1 - \xi_3\hat{\sigma}(t), \quad \hat{\sigma}(0) = \hat{\sigma}_0, \tag{2.69}$$

where $\hat{\mu}(t) \in \mathbb{R}$, $t \geq 0$, is the estimate of $\mu \triangleq q(1+q)^{-1} \in \mathbb{R}$ that depends on the sensor uncertainty q, $\hat{W}(t) \in \mathbb{R}^{p\times m}$, $t \geq 0$, is the estimate of the parametric uncertainty W, $\hat{\sigma}(t) \in \mathbb{R}$, $t \geq 0$, is the estimate of the unknown bound $\bar{\sigma}$, and $\gamma \in \mathbb{R}$, $\eta \in \mathbb{R}$, $\nu \in \mathbb{R}$, $\xi_1 \in \mathbb{R}$, $\xi_2 \in \mathbb{R}$, and $\xi_3 \in \mathbb{R}$ are positive design gains.

Next, using (2.61) and (2.62), (2.63) can be equivalently written as

$$\dot{x}(t) = Ax(t) + B\Big[u(t) + W^{\mathrm{T}}\varphi(\tilde{x}(t)) + \sigma(\tilde{x}(t))\Big], \quad x(0) = x_0, \quad t \geq 0. \tag{2.70}$$

Now, define $\tilde{\mu}(t) \triangleq \mu - \hat{\mu}(t)$, $t \geq 0$, $\tilde{W}(t) \triangleq W - \hat{W}(t)$, $t \geq 0$, $\tilde{\sigma}(t) \triangleq \bar{\sigma} - \hat{\sigma}(t)$, $t \geq 0$, and $\lambda \triangleq \big(1+q\big)^{-1}$. Since $q > -1$, note that μ and λ are well-defined and $\lambda > 0$. Next, using $qx = \mu\tilde{x}$, (2.60), (2.64), and (2.66), it follows from (2.70) that

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + BK\tilde{x}(t) + Bv(t) + B\Big[W^{\mathrm{T}}\varphi(\tilde{x}(t)) + \sigma(\tilde{x}(t))\Big] \\
&= Ax(t) + BKx(t) + BKqx(t) + Bv(t) + B\Big[W^{\mathrm{T}}\varphi(\tilde{x}(t)) + \sigma(\tilde{x}(t))\Big] \\
&= A_{\mathrm{r}}x(t) + BK\mu\tilde{x}(t) - \hat{\mu}(t)BK\tilde{x}(t) - B\hat{W}^{\mathrm{T}}(t)\varphi(\tilde{x}(t))
\end{aligned}$$

$$
\begin{aligned}
&\quad - B\hat{\sigma}(t)\mathrm{sgn_v}(B^{\mathrm{T}}P\tilde{x}(t)) + B\Big[W^{\mathrm{T}}\varphi(\tilde{x}(t)) + \sigma(\tilde{x}(t))\Big] \\
&= A_{\mathrm{r}}x(t) + \tilde{\mu}(t)BK\tilde{x}(t) + B\tilde{W}^{\mathrm{T}}(t)\varphi(\tilde{x}(t)) + B(\sigma(\tilde{x}(t)) \\
&\quad - \hat{\sigma}(t)\mathrm{sgn_v}(B^{\mathrm{T}}P\tilde{x}(t))), \quad x(0) = x_0, \quad t \geq 0. \qquad (2.71)
\end{aligned}
$$

Theorem 2.3. Consider the dynamical system given by (2.63) with sensor and actuator attacks given by (2.60) and (2.61), respectively. Then, with the controller $\mathcal{G}_{\mathrm{c}}$ given by (2.64), the corrective signal $v(t)$, $t \geq 0$, given by (2.66), and adaptive laws given by (2.67)–(2.69), the closed-loop system given by (2.67)–(2.69) and (2.71) satisfies

$$
\limsup_{t\to\infty} \|x(t)\|^2 \leq \frac{c_0}{c_1\lambda_{\min}(P)}, \qquad (2.72)
$$

where $c_0 \triangleq \gamma^{-1}\xi_1\lambda\mu^2 + \eta^{-1}\xi_2\lambda\mathrm{tr}(W^{\mathrm{T}}W) + \nu^{-1}\xi_3\lambda\bar{\sigma}^2$ and $c_1 \triangleq \min\left\{\frac{\lambda_{\min}(R)}{\lambda_{\max}(P)}, \xi_1, \xi_2, \xi_3\right\}$. In addition, the adaptive estimates $\hat{\mu}(t)$, $t \geq 0$, $\hat{W}(t)$, $t \geq 0$, and $\hat{\sigma}(t)$, $t \geq 0$, are ultimately uniformly bounded.

Proof. To show ultimate boundedness of the closed-loop system, consider the Lyapunov-like function given by

$$
V\big(x, \tilde{\mu}, \tilde{W}, \tilde{\sigma}\big) = x^{\mathrm{T}}Px + \frac{\lambda}{\gamma}\tilde{\mu}^2 + \frac{\lambda}{\eta}\mathrm{tr}(\tilde{W}^{\mathrm{T}}\tilde{W}) + \frac{\lambda}{\nu}\tilde{\sigma}^2, \qquad (2.73)
$$

where P satisfies (2.65). Now, the derivative of (2.73) along the closed-loop system trajectories (2.67)–(2.69) is given by

$$
\begin{aligned}
\dot{V}\big(x, \tilde{\mu}, \tilde{W}, \tilde{\sigma}\big) &= x^{\mathrm{T}}PA_{\mathrm{r}}x + x^{\mathrm{T}}A_{\mathrm{r}}^{\mathrm{T}}Px + 2\tilde{\mu}x^{\mathrm{T}}PBK\tilde{x} + 2x^{\mathrm{T}}PB\tilde{W}^{\mathrm{T}}\varphi(\tilde{x}) \\
&\quad + 2x^{\mathrm{T}}PB(\sigma(\tilde{x}) - \hat{\sigma}\mathrm{sgn_v}(B^{\mathrm{T}}P\tilde{x})) \\
&\quad + 2\frac{\lambda}{\gamma}\tilde{\mu}[-\gamma\tilde{x}^{\mathrm{T}}PBK\tilde{x} + \xi_1\hat{\mu}] + 2\frac{\lambda}{\eta}\mathrm{tr}[\tilde{W}^{\mathrm{T}}(-\eta\varphi(\tilde{x})\tilde{x}^{\mathrm{T}}PB + \xi_2\hat{W})] \\
&\quad + 2\frac{\lambda}{\nu}\tilde{\sigma}[-\nu\|\tilde{x}^{\mathrm{T}}BP\|_1 + \xi_3\hat{\sigma}], \quad (x, \tilde{\mu}, \tilde{W}, \tilde{\sigma}) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{p\times m} \times \mathbb{R}. \quad (2.74)
\end{aligned}
$$

Noting that $2x^{\mathrm{T}}PB\tilde{W}^{\mathrm{T}}\varphi(\tilde{x}) = 2\mathrm{tr}[\tilde{W}^{\mathrm{T}}\varphi(\tilde{x})x^{\mathrm{T}}PB]$, it follows from (2.74) that

$$
\dot{V}\big(x, \tilde{\mu}, \tilde{W}, \tilde{\sigma}\big) = x^{\mathrm{T}}\left[A_{\mathrm{r}}^{\mathrm{T}}P + PA_{\mathrm{r}}\right]x + 2\tilde{\mu}x^{\mathrm{T}}PBK\tilde{x}
$$

$$
\begin{aligned}
&+ 2\frac{\lambda}{\gamma}\tilde{\mu}[-\gamma\tilde{x}^{\mathrm{T}}PBK\tilde{x} + \xi_1\hat{\mu}] + 2\mathrm{tr}[\tilde{W}^{\mathrm{T}}\varphi(\tilde{x})x^{\mathrm{T}}PB] \\
&+ 2\frac{\lambda}{\eta}\mathrm{tr}[\tilde{W}^{\mathrm{T}}(-\eta\varphi(\tilde{x})\tilde{x}^{\mathrm{T}}PB + \xi_2\hat{W})] \\
&+ 2x^{\mathrm{T}}PB(\sigma(\tilde{x}) - \hat{\sigma}\mathrm{sgn_v}(B^{\mathrm{T}}P\tilde{x})) + 2\frac{\lambda}{\nu}\tilde{\sigma}[-\nu\|\tilde{x}^{\mathrm{T}}BP\|_1 + \xi_3\hat{\sigma}], \\
&\qquad\qquad (x, \tilde{\mu}, \tilde{W}, \tilde{\sigma}) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{p\times m} \times \mathbb{R}. \qquad (2.75)
\end{aligned}
$$

Next, using $\tilde{x}^{\mathrm{T}}PB\sigma(\tilde{x}) \le \|\tilde{x}^{\mathrm{T}}PB\|\bar{\sigma} \le \|\tilde{x}^{\mathrm{T}}PB\|_1\bar{\sigma}$ and $\tilde{x}^{\mathrm{T}}PB\mathrm{sgn_v}(B^{\mathrm{T}}P\tilde{x}) = \|\tilde{x}^{\mathrm{T}}PB\|_1$, it follows that

$$
\begin{aligned}
2x^{\mathrm{T}}PB[\sigma(\tilde{x}) - \hat{\sigma}\mathrm{sgn_v}(B^{\mathrm{T}}P\tilde{x})] &\le 2\lambda\|\tilde{x}^{\mathrm{T}}PB\|_1\overline{\sigma} - 2\lambda\|\tilde{x}^{\mathrm{T}}PB\|_1\hat{\sigma} \\
&= 2\lambda\|\tilde{x}^{\mathrm{T}}PB\|_1\tilde{\sigma}, \quad x \in \mathbb{R}^n. \qquad (2.76)
\end{aligned}
$$

Now, using (2.65), it follows from (2.75) that

$$
\begin{aligned}
\dot{V}\big(x, \tilde{\mu}, \tilde{W}, \tilde{\sigma}\big) \le{}& -x^{\mathrm{T}}Rx + 2\tilde{\mu}\lambda\tilde{x}^{\mathrm{T}}PBK\tilde{x} + 2\frac{\lambda}{\gamma}\tilde{\mu}[-\gamma\tilde{x}^{\mathrm{T}}PBK\tilde{x} + \xi_1\hat{\mu}] \\
&+ 2\mathrm{tr}[\tilde{W}^{\mathrm{T}}\lambda\varphi(\tilde{x})\tilde{x}^{\mathrm{T}}PB] + 2\frac{\lambda}{\eta}\mathrm{tr}[\tilde{W}^{\mathrm{T}}(-\eta\varphi(\tilde{x})\tilde{x}^{\mathrm{T}}PB + \xi_2\hat{W})] \\
&+ 2\|\tilde{x}^{\mathrm{T}}BP\|_1\tilde{\sigma}\lambda + 2\frac{\lambda}{\nu}\tilde{\sigma}[-\nu\|\tilde{x}^{\mathrm{T}}BP\|_1 + \xi_3\hat{\sigma}] \\
\le{}& -x^{\mathrm{T}}Rx + 2\frac{\lambda}{\gamma}\xi_1\tilde{\mu}\hat{\mu} + 2\frac{\lambda}{\eta}\xi_2\mathrm{tr}(\tilde{W}^{\mathrm{T}}\hat{W}) \\
&+ 2\frac{\lambda}{\nu}\xi_3\tilde{\sigma}\hat{\sigma}, \quad (x, \tilde{\mu}, \tilde{W}, \tilde{\sigma}) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{p\times m} \times \mathbb{R}. \qquad (2.77)
\end{aligned}
$$

Next, writing $2\frac{\lambda}{\gamma}\xi_1\tilde{\mu}\hat{\mu}$ and $2\frac{\lambda}{\eta}\xi_2\mathrm{tr}(\tilde{W}^{\mathrm{T}}\hat{W})$ in (2.77), respectively, as

$$
\begin{aligned}
2\frac{\lambda}{\gamma}\xi_1\tilde{\mu}\hat{\mu} = 2\frac{\lambda}{\gamma}\xi_1\tilde{\mu}(\mu - \tilde{\mu}) &= 2\frac{\lambda}{\gamma}\xi_1\tilde{\mu}\mu - 2\frac{\lambda}{\gamma}\xi_1\tilde{\mu}^2 \\
&\le \frac{\lambda}{\gamma}\xi_1\tilde{\mu}^2 + \frac{\lambda}{\gamma}\xi_1\mu^2 - 2\frac{\lambda}{\gamma}\xi_1\tilde{\mu}^2 \\
&= -\frac{\lambda}{\gamma}\xi_1\tilde{\mu}^2 + \frac{\lambda}{\gamma}\xi_1\mu^2, \qquad (2.78)
\end{aligned}
$$

where $\frac{\lambda}{\gamma}\xi_1\mu^2$ is a constant, and

$$
2\frac{\lambda}{\eta}\xi_2\mathrm{tr}(\tilde{W}^{\mathrm{T}}\hat{W}) \le -\frac{\lambda}{\eta}\xi_2\mathrm{tr}(\tilde{W}^{\mathrm{T}}\tilde{W}) + \frac{\lambda}{\eta}\xi_2\mathrm{tr}(W^{\mathrm{T}}W), \qquad (2.79)
$$

$$2\frac{\lambda}{\nu}\xi_3\tilde{\sigma}\hat{\sigma} \le -\frac{\lambda}{\nu}\xi_3\tilde{\sigma}^2 + \frac{\lambda}{\nu}\xi_3\bar{\sigma}^2, \tag{2.80}$$

it follows from (2.77)–(2.80) that

$$\begin{aligned}
\dot{V}\big(x,\tilde{\mu},\tilde{W},\tilde{\sigma}\big) &\le -x^{\mathrm{T}}Rx - \gamma^{-1}\lambda\xi_1\tilde{\mu}^2 - \eta^{-1}\lambda\xi_2\mathrm{tr}(\tilde{W}^{\mathrm{T}}\tilde{W}) - \nu^{-1}\lambda\xi_3\tilde{\sigma}^2 + \gamma^{-1}\xi_1\lambda\mu^2 \\
&\quad + \eta^{-1}\xi_2\lambda\mathrm{tr}(W^{\mathrm{T}}W) + \nu^{-1}\xi_3\lambda\bar{\sigma}^2 \\
&\le -\frac{\lambda_{\min}(R)}{\lambda_{\max}(P)}x^{\mathrm{T}}Px - \gamma^{-1}\lambda\xi_1\tilde{\mu}^2 - \eta^{-1}\lambda\xi_2\mathrm{tr}(\tilde{W}^{\mathrm{T}}\tilde{W}) \\
&\quad - \nu^{-1}\lambda\xi_3\tilde{\sigma}^2 + c_0 \\
&\le -c_1V\big(x,\tilde{\mu},\tilde{W},\tilde{\sigma}\big) + c_0, \quad (x,\tilde{\mu},\tilde{W},\tilde{\sigma}) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{p\times m} \times \mathbb{R},
\end{aligned} \tag{2.81}$$

where $c_0 \triangleq \frac{\lambda}{\gamma}\xi_1\mu^2 + \frac{\lambda}{\eta}\xi_2\mathrm{tr}(W^{\mathrm{T}}W) + \frac{\lambda}{\nu}\xi_3\bar{\sigma}^2$ and $c_1 \triangleq \min\left\{\frac{\lambda_{\min}(R)}{\lambda_{\max}(P)}, \xi_1, \xi_2, \xi_3\right\}$. Thus it follows from (2.81) that

$$0 \le V\big(x(t),\tilde{\mu}(t),\tilde{W}(t),\tilde{\sigma}(t)\big) \le V\big(x(0),\tilde{\mu}(0),\tilde{W}(0),\tilde{\sigma}(0)\big)e^{-c_1 t} + \frac{c_0}{c_1}, \quad t \ge 0, \tag{2.82}$$

and hence, all the signals of the closed-loop system are uniformly ultimately bounded. Furthermore, noting that

$$\limsup_{t\to\infty} x^{\mathrm{T}}(t)Px(t) \le \limsup_{t\to\infty} V\big(x(t),\tilde{\mu}(t),\tilde{W}(t),\tilde{\sigma}(t)\big) \le \frac{c_0}{c_1}, \tag{2.83}$$

it follows that

$$\limsup_{t\to\infty} \|x(t)\|^2 \le \frac{c_0}{c_1\lambda_{\min}(P)}, \tag{2.84}$$

which implies that the trajectory of the closed-loop system associated with the plant dynamics is uniformly ultimately bounded. □

Theorem 2.3 assumes that $q > -1$. This assumption implies that $\lambda > 0$. As long as the sign of λ is known, Theorem 2.3 can be used to address the case where $\lambda < 0$. The assumption $q > -1$ can be relaxed by utilizing tools from [73] that can allow λ to have any sign as long as $q \neq -1$ under the assumption that its sign is a priori known.

In arriving at Theorem 2.3 we assumed that $\|\sigma(\tilde{x})\| \le \bar{\sigma}$, $\tilde{x} \in \mathbb{R}^n$, where $\bar{\sigma} > 0$ is unknown. Alternatively, we can assume that $\sigma(\tilde{x})$ satisfies the Lipschitz condition $\|\sigma(\tilde{x})\| \le \bar{\sigma}\|\tilde{x}\|$,

$\tilde{x} \in \mathbb{R}^n$, where $\bar{\sigma} > 0$ is an *unknown* Lipschitz constant. In this case, it can be shown that Theorem 2.3 holds with (2.66) replaced by

$$v(t) = -\hat{\mu}(t)K\tilde{x}(t) - \hat{W}^{\mathrm{T}}(t)\varphi(\tilde{x}(t)) - \hat{\sigma}(t)\|\tilde{x}(t)\|\mathrm{sgn}_{\mathrm{v}}(B^{\mathrm{T}}P\tilde{x}(t)) \tag{2.85}$$

and with (2.69) replaced by

$$\dot{\hat{\sigma}}(t) = \nu\|\tilde{x}(t)\|\|\tilde{x}^{\mathrm{T}}(t)PB\|_1 - \xi_3\hat{\sigma}(t), \quad \hat{\sigma}(0) = \hat{\sigma}_0, \quad t \geq 0. \tag{2.86}$$

Note that the controller $u(t)$, $t \geq 0$, given by (2.64) is discontinuous because of the presence of the signum function $\mathrm{sgn}_{\mathrm{v}}(\cdot)$ in the controller architecture. This discontinuity can lead to a chattering phenomenon, which is undesirable in practice. In order to reduce or eliminate the chattering effect, a smooth function can be implemented instead of the signum function [109]; that is, we replace $\mathrm{sgn}_{\mathrm{v}}(\cdot)$ by $\tanh_{\mathrm{v}}(\cdot)$, where, for $y \in \mathbb{R}^n$, $\tanh_{\mathrm{v}}(y) \triangleq [\tanh(y_1), \ldots, \tanh(y_n)]^{\mathrm{T}}$. Note that ([109])

$$0 \leq |\alpha| - \alpha \tanh\left(\frac{\alpha}{\varepsilon}\right) \leq k_0\varepsilon, \qquad \alpha \in \mathbb{R}, \tag{2.87}$$

where $\varepsilon > 0$ is a design constant and k_0 satisfies $k_0 = e^{-(k_0+1)}$ with $k_0 = 0.2785$. Thus, we modify (2.66) as

$$v(t) = -\hat{\mu}(t)K\tilde{x}(t) - \hat{W}^{\mathrm{T}}(t)\varphi(\tilde{x}(t)) - \hat{\sigma}(t)\tanh_{\mathrm{v}}\left(\frac{B^{\mathrm{T}}P\tilde{x}(t)\hat{\sigma}(t)}{\varepsilon}\right). \tag{2.88}$$

In this case, (2.76) becomes

$$\begin{aligned}
&2x^{\mathrm{T}}PB\left[\sigma(x) - \hat{\sigma}\tanh_{\mathrm{v}}\left(\frac{B^{\mathrm{T}}P\tilde{x}\hat{\sigma}}{\varepsilon}\right)\right] \\
&\quad \leq 2\|\tilde{x}^{\mathrm{T}}PB\|_1\bar{\sigma}\lambda - 2\tilde{x}^{\mathrm{T}}PB\hat{\sigma}\lambda\tanh_{\mathrm{v}}\left(\frac{B^{\mathrm{T}}P\tilde{x}\hat{\sigma}}{\varepsilon}\right) \\
&\quad = 2\|\tilde{x}^{\mathrm{T}}PB\|_1(\hat{\sigma} + \tilde{\sigma})\lambda - 2\tilde{x}^{\mathrm{T}}PB\hat{\sigma}\lambda\tanh_{\mathrm{v}}\left(\frac{B^{\mathrm{T}}P\tilde{x}\hat{\sigma}}{\varepsilon}\right) \\
&\quad \leq 2\|\tilde{x}^{\mathrm{T}}PB\|_1(|\hat{\sigma}| + \tilde{\sigma})\lambda - 2\tilde{x}^{\mathrm{T}}PB\hat{\sigma}\lambda\tanh_{\mathrm{v}}\left(\frac{B^{\mathrm{T}}P\tilde{x}\hat{\sigma}}{\varepsilon}\right) \\
&\quad = 2\|\tilde{x}^{\mathrm{T}}PB\|_1\tilde{\sigma}\lambda + \sum_{i=1}^{m} 2\lambda\left[|(\tilde{x}^{\mathrm{T}}PB)_i\hat{\sigma} - (\tilde{x}^{\mathrm{T}}PB)_i\hat{\sigma}\tanh\left(\frac{(\tilde{x}^{\mathrm{T}}PB)_i\hat{\sigma}}{\varepsilon}\right)\right]
\end{aligned}$$

$$\leq 2\|\tilde{x}^{\mathrm{T}}PB\|_1\tilde{\sigma}\lambda + 2m\lambda k_0\varepsilon. \tag{2.89}$$

Now, it follows from (2.81) that

$$\begin{aligned}
\dot{V}\big(x,\tilde{\mu},\tilde{W},\tilde{\sigma}\big) &\leq -x^{\mathrm{T}}Rx - \gamma^{-1}\lambda\xi_1\tilde{\mu}^2 - \eta^{-1}\lambda\xi_2\mathrm{tr}(\tilde{W}^{\mathrm{T}}\tilde{W}) - \nu^{-1}\lambda\xi_3\tilde{\sigma}^2 + \gamma^{-1}\xi_1\lambda\mu^2 \\
&\quad + \eta^{-1}\xi_2\lambda\mathrm{tr}(W^{\mathrm{T}}W) + \nu^{-1}\xi_3\lambda\bar{\sigma}^2 + 2m\lambda k_0\varepsilon \\
&\leq -\frac{\lambda_{\min}(R)}{\lambda_{\max}(P)}x^{\mathrm{T}}Px - \gamma^{-1}\lambda\xi_1\tilde{\mu}^2 - \eta^{-1}\lambda\xi_2\mathrm{tr}(\tilde{W}^{\mathrm{T}}\tilde{W}) - \nu^{-1}\lambda\xi_3\tilde{\sigma}^2 + c_2 \\
&\leq -c_1V\big(x,\tilde{\mu},\tilde{W},\tilde{\sigma}\big) + c_2, \quad (x,\tilde{\mu},\tilde{W},\tilde{\sigma}) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{p\times m} \times \mathbb{R},
\end{aligned} \tag{2.90}$$

where $c_2 \triangleq \gamma^{-1}\xi_1\lambda\mu^2 + \eta^{-1}\xi_2\lambda\mathrm{tr}(W^{\mathrm{T}}W) + \nu^{-1}\xi_3\lambda\bar{\sigma}^2 + 2m\lambda k_0\varepsilon = c_0 + 2m\lambda k_0\varepsilon$ and $c_1 = \min\left\{\frac{\lambda_{\min}(R)}{\lambda_{\max}(P)}, \xi_1, \xi_2, \xi_3\right\}$. Using similar arguments as in the proof of Theorem 2.3, it can be shown that

$$\limsup_{t\to\infty} \|x(t)\|^2 \leq \frac{c_2}{c_1\lambda_{\min}(P)}, \tag{2.91}$$

which is identical to the result of Theorem 2.3 with the only difference being that c_0 is replaced by c_2.

In practice, the constant $\frac{c_2}{c_1\lambda_{\min}(P)}$ in (2.91) should be made small so that the trajectory of the system can be regulated as close to the equilibrium as possible. This can be achieved by taking a large value for c_1 and a small value for c_2. Note that since $c_2 = \gamma^{-1}\xi_1\lambda\mu^2 + \eta^{-1}\xi_2\lambda\mathrm{tr}(W^{\mathrm{T}}W) + \nu^{-1}\xi_3\lambda\bar{\sigma}^2 + 2m\lambda k_0\varepsilon$ and $c_1 = \min\left\{\frac{\lambda_{\min}(R)}{\lambda_{\max}(P)}, \xi_1, \xi_2, \xi_3\right\}$, the value of c_1 can be made large by choosing large values for $\frac{\lambda_{\min}(R)}{\lambda_{\max}(P)}$, ξ_1, ξ_2, and ξ_3. Alternatively, the value of c_2 can be made small by choosing small values for $\gamma^{-1}\xi_1$, $\eta^{-1}\xi_2$, $\nu^{-1}\xi_3$, and $2m\overline{\lambda}k_0\varepsilon$, which can be achieved by choosing large values for γ, η, and ν, and a small value for ε. However, since γ, η, and ν are design gain parameters used in the adaptive laws (2.67)–(2.69), selecting large values of these parameters can introduce transient oscillations in the update law estimates of $\hat{\mu}$, $\hat{W}$, and $\hat{\sigma}$, and hence, in the control signal $u(t)$, $t \geq 0$. This can be remedied by adding a modification term in the update laws to filter out the high frequency content in the control signal while preserving uniform ultimate boundedness. For details of a similar approach, see [155].

2.7. Application to Platooning Connected Vehicles

Table 2.1: system parameters

Parameter	Meaning	Value
$v_{\rm m}$[m/s]	Maximum velocity	30
$h_{\rm s}$[m]	Stop headway	5
$h_{\rm g}$[m]	Headway for maximum velocity	35
h^*[m]	Desired headway	20
v_1[m]	Desired velocity	15
α_2	Headway gain of vehicle 2	0.15
β_2	Relative velocity gain of vehicle 2	0.25
α_3	Headway gain of vehicle 3	0.15
β_3	Relative velocity gain of vehicle 3	0.2
α_4	Headway gain of vehicle 4	0.25
β_4	Relative velocity gain of vehicle 4	0.25

To illustrate the key ideas presented in this section, we simulate a platoon of 4+1 vehicles where the 4 forward vehicles are human-driven with different human parameter values. The system parameters are given in Table 2.1. To design the proposed adaptive control law and corrective signal given by (2.67)–(2.69), we set $\gamma = 0.8$, $\xi = 0.8$, $\eta = 0.8$, $\nu = 0.8$, $\xi_1 = 2$, $\xi_2 = 2$, $\xi_3 = 2$, $\varepsilon = 0.001$, and $R = 3I_8$. In this case, P satisfying (2.65) is given by

$$P = \begin{bmatrix} 23.6224 & -6.0540 & 7.2074 & -20.9879 & -1.3591 & -2.6040 & -0.2206 & -1.3545 \\ -6.0540 & 80.3395 & 22.9915 & 12.0654 & 7.7603 & -8.2184 & 4.0323 & -11.8700 \\ 7.2074 & 22.9915 & 15.7720 & -4.1640 & 3.8360 & -5.2957 & 2.0694 & -6.7856 \\ -20.9879 & 12.0654 & -4.1640 & 57.3733 & 15.5864 & -0.0798 & 7.7480 & -16.0999 \\ -1.3591 & 7.7603 & 3.8360 & 15.5864 & 11.9348 & -0.8313 & 5.7701 & -12.4968 \\ -2.6040 & -8.2184 & -5.2957 & -0.0798 & -0.8313 & 17.6689 & 6.7693 & -8.4999 \\ -0.2206 & 4.0323 & 2.0694 & 7.7480 & 5.7701 & 6.7693 & 10.4745 & -15.0000 \\ -1.3545 & -11.8700 & -6.7856 & -16.0999 & -12.4968 & -8.4999 & -15.0000 & 36.0060 \end{bmatrix}. \tag{2.92}$$

The system performance of the controller without corrective signal is depicted in Figures 2.7.1 and 2.7.2 for position and velocity regulation, respectively. It can be seen that due to the

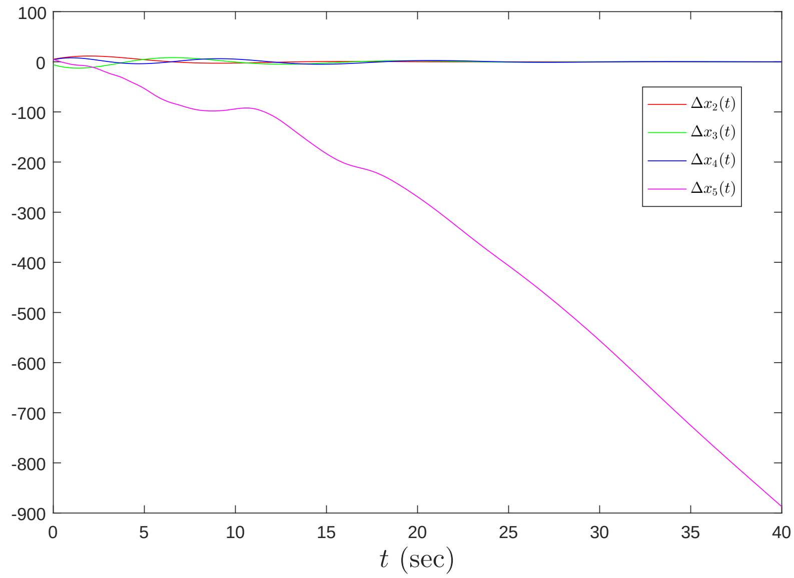


Figure 2.7.1: Relative distance of the connected vehicles in the presence of sensor and actuator attacks without the proposed corrective signal.

presence of sensor and actuator attacks the aft vehicle fails to maintain the desired formation. The system performance of the controller given by (2.64) with the proposed corrective signal is depicted in Figures 2.7.3 and 2.7.4 for position and velocity regulation, respectively. The proposed adaptive control architecture achieves satisfactory system performance in the face of sensor and actuator attacks.

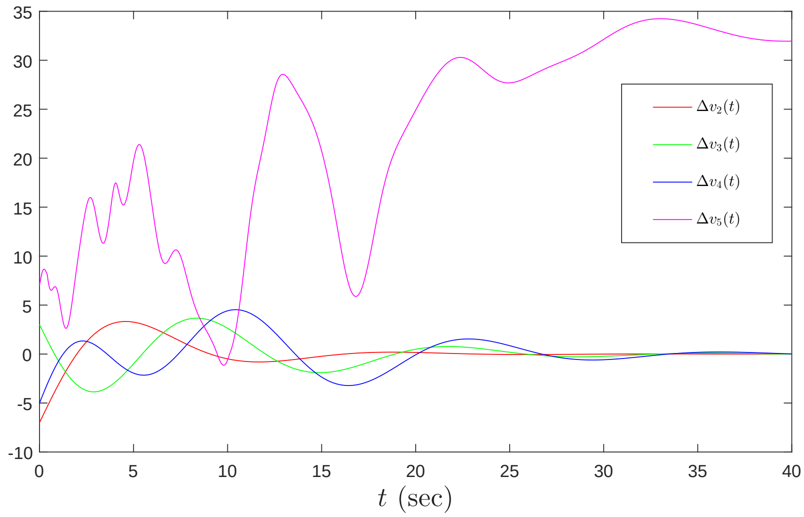


Figure 2.7.2: Relative velocity of the connected vehicles in the presence of sensor and actuator attacks without the proposed corrective signal.

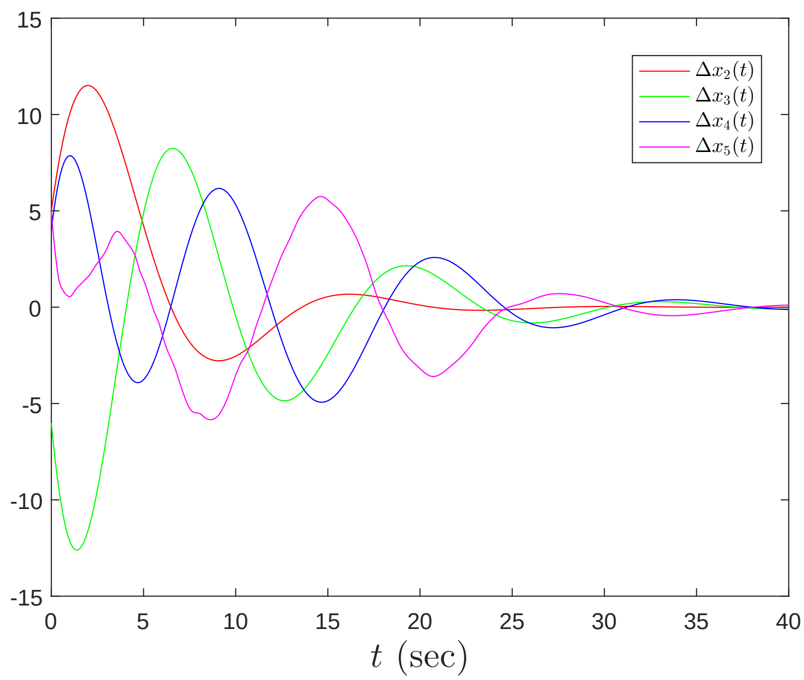


Figure 2.7.3: Relative distance of the connected vehicles in the presence of sensor and actuator attacks with the proposed corrective signal given by (2.88), (2.67)–(2.69).

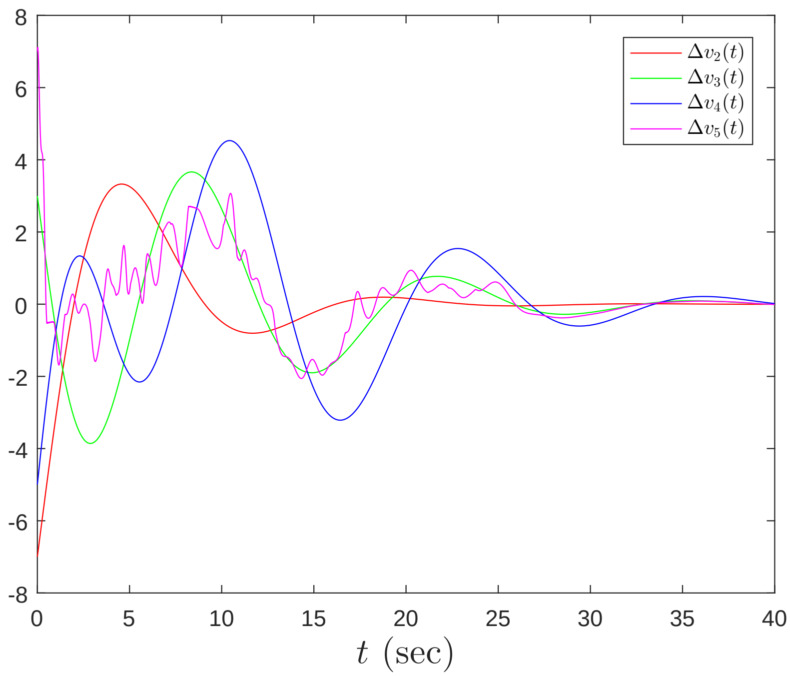


Figure 2.7.4: Relative velocity of the connected vehicles in the presence of sensor and actuator attacks with the proposed corrective signal given by (2.88), (2.67)–(2.69).

Chapter 3

Adaptive Control for Cyber-Physical System Security in the Face of Sensor and Actuator Attacks and Stochastic Disturbances

3.1. Introduction

In this chapter, we extend the results in Chapter 2 to develop new adaptive control architectures that can foil malicious sensor and actuator attacks in the face of exogenous stochastic disturbances. Specifically, to address the dynamic relationships of sequences of random events between system-environment interactions, we extend our recent work on cyber-physical security and safety [67, 156] to develop an adaptive controller for mitigating time-invariant, state-dependent sensor and actuator attacks subject to random disturbances modeled as Markov processes. Furthermore, we show that the proposed controller guarantees uniform ultimate boundedness in probability of the closed-loop stochastic dynamical system in a mean-square sense.

The proposed controller is composed of two components, namely a nominal controller and an additive corrective signal. It is assumed that the nominal controller has been already designed and implemented to achieve a desired closed-loop nominal performance. Using the nominal controller, an additive adaptive corrective signal is designed and added to the output of the nominal controller in order to suppress the effects of the sensor and actuator attacks. Thus, the proposed controller is modular in the sense that there is no need to redesign the nominal controller in the proposed framework; only the adaptive corrective signal is designed using the available information from the nominal controller and the system.

3.2. Notation, Definitions, and Mathematical Preliminaries

In this section, we establish notation, definitions, and develop mathematical preliminaries necessary for developing the remaining results in this proposal. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ denotes the set of positive real numbers, $\overline{\mathbb{R}}_+$ denotes the set of nonnegative numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, and $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices. We write $\mathcal{B}_\varepsilon(x)$ for the *open ball centered* at x with *radius* ε, $\|\cdot\|$ for the Euclidean vector norm or an induced matrix norm (depending on context), $\|\cdot\|_{\mathrm{F}}$ for the Frobenius matrix norm, A^{T} for the transpose of the matrix A, and I_n or I for the $n \times n$ identity matrix. Furthermore, $\mathfrak{B}^n$ denotes the σ-algebra of Borel sets in $\mathcal{D} \subseteq \mathbb{R}^n$ and $\mathfrak{S}$ denotes a σ-algebra generated on a set $\mathcal{S} \subseteq \mathbb{R}^n$.

We define a complete probability space as $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω denotes the sample space, $\mathcal{F}$ denotes a σ-algebra, and $\mathbb{P}$ defines a probability measure on the σ-algebra $\mathcal{F}$; that is, $\mathbb{P}$ is a nonnegative countably additive set function on $\mathcal{F}$ such that $\mathbb{P}(\Omega) = 1$ [5]. Furthermore, we assume that $w(\cdot)$ is a standard d-dimensional Wiener process defined by $(w(\cdot), \Omega, \mathcal{F}, \mathbb{P}^{w_0})$, where $\mathbb{P}^{w_0}$ is the classical Wiener measure [94, p. 10], with a continuous-time filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by the Wiener process $w(t)$ up to time t. We denote by $\mathcal{G}$ a stochastic dynamical system generating a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ adapted to the stochastic process $x : \overline{\mathbb{R}}_+ \times \Omega \to \mathcal{D}$ on $(\Omega, \mathcal{F}, \mathbb{P}^{x_0})$ satisfying $\mathcal{F}_\tau \subset \mathcal{F}_t$, $0 \leq \tau < t$, such that $\{\omega \in \Omega : x(t,\omega) \in \mathcal{B}\} \in \mathcal{F}_t$, $t \geq 0$, for all Borel sets $\mathcal{B} \subset \mathbb{R}^n$ contained in the Borel σ-algebra $\mathfrak{B}^n$. Here we use the notation $x(t)$ to represent the stochastic process $x(t,\omega)$ omitting its dependence on ω.

We denote the set of equivalence classes of measurable, integrable, and square-integrable $\mathbb{R}^n$ or $\mathbb{R}^{n\times m}$ (depending on context) valued random processes on $(\Omega, \mathcal{F}, \mathbb{P})$ over the semi-infinite parameter space $[0,\infty)$ by $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, respectively, where the equivalence relation is the one induced by $\mathbb{P}$-almost-sure equality. In particular, elements of $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ take finite values $\mathbb{P}$-almost surely (a.s.). Hence, depending on the context, $\mathbb{R}^n$ will denote either the set of $n \times 1$ real variables or the subspace of $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ comprising of $\mathbb{R}^n$ random processes that are constant almost surely. All inequalities and equalities involving random processes on $(\Omega, \mathcal{F}, \mathbb{P})$ are to be understood to hold $\mathbb{P}$-almost surely. Furthermore, $\mathbb{E}[\,\cdot\,]$ and $\mathbb{E}^{x_0}[\,\cdot\,]$ denote, respectively, the expectation with respect to the probability measure $\mathbb{P}$ and with respect to the classical Wiener measure $\mathbb{P}^{x_0}$.

Finally, we write $\operatorname{tr}(\cdot)$ for the trace operator, $(\cdot)^{-1}$ for the inverse operator, $V'(x) \triangleq \frac{\partial V(x)}{\partial x}$ for the Fréchet derivative of V at x, $V''(x) \triangleq \frac{\partial^2 V(x)}{\partial x^2}$ for the Hessian of V at x, and $\mathcal{H}_n$ for the Hilbert space of random vectors $x \in \mathbb{R}^n$ with finite average power, that is, $\mathcal{H}_n \triangleq \{x : \Omega \to \mathbb{R}^n : \mathbb{E}[x^{\mathrm{T}}x] < \infty\}$. For an open set $\mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{H}_n^{\mathcal{D}} \triangleq \{x \in \mathcal{H}_n : x : \Omega \to \mathcal{D}\}$ denotes the set of all the random vectors in $\mathcal{H}_n$ induced by $\mathcal{D}$. Similarly, for every $x_0 \in \mathbb{R}^n$, $\mathcal{H}_n^{x_0} \triangleq \{x \in \mathcal{H}_n : x \overset{\text{a.s.}}{=} x_0\}$. Furthermore, C^2 denotes the space of real-valued functions $V : \mathcal{D} \to \mathbb{R}$ that are two-times continuously differentiable with respect to $x \in \mathcal{D} \subseteq \mathbb{R}^n$. Finally, we write $x(t) \overset{\text{a.s.}}{\to} \mathcal{M}$ as $t \to \infty$ to denote that $x(t)$ approaches the set $\mathcal{M}$ almost surely, that is, for every $\varepsilon > 0$ there exists finite stopping time $T > 0$ such that $\operatorname{dist}(x(t), \mathcal{M}) < \varepsilon$ for all $t > T$, where $\operatorname{dist}(p, \mathcal{M}) \triangleq \inf_{x \in \mathcal{M}} \|p - x\|$.

Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by

$$\mathrm{d}x(t) = f(x(t))\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(t_0) \overset{\text{a.s.}}{=} x_0, \quad t \geq t_0, \tag{3.1}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$ is a $\mathcal{F}_t$-measurable random state vector, $x(t_0) \in \mathcal{H}_n^{x_0}$, $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set with $0 \in \mathcal{D}$, $w(t)$ is a d-dimensional independent standard Wiener process (i.e., Brownian motion) defined on a complete filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{P})$, $x(t_0)$ is independent of $(w(t) - w(t_0)), t \geq t_0$, and $f : \mathcal{D} \to \mathbb{R}^n$ and $D : \mathcal{D} \to \mathbb{R}^{n \times d}$ are continuous functions and satisfy $f(x_{\mathrm{e}}) = 0$ and $D(x_{\mathrm{e}}) = 0$ for some $x_{\mathrm{e}} \in \mathcal{D}$. An *equilibrium point* of (3.1) is a point $x_{\mathrm{e}} \in \mathcal{D}$ such that $f(x_{\mathrm{e}}) = 0$ and $D(x_{\mathrm{e}}) = 0$. It is easy to see that x_{e} is an equilibrium point of (3.1) if and only if the constant stochastic process $x(\cdot) \overset{\text{a.s.}}{=} x_{\mathrm{e}}$ is a solution of (3.1). We denote the set of equilibrium points of (3.1) by $\mathcal{E} \triangleq \{\omega \in \Omega : x(t,\omega) = x_{\rm e}\} = \{x_{\rm e} \in \mathcal{D} : f(x_{\rm e}) = 0 \text{ and } D(x_{\rm e}) = 0\}$.

The filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P})$ is clearly a real vector space with addition and scalar multiplication defined componentwise and pointwise. A $\mathbb{R}^n$-valued stochastic process $x : [t_0, \tau] \times \Omega \to \mathcal{D}$ is said to be a *solution* of (3.1) on the time interval $[t_0, \tau]$ with initial condition $x(t_0) \overset{\text{a.s.}}{=} x_0$ if $x(\cdot)$ is *progressively measurable* (i.e., $x(\cdot)$ is nonanticipating and measurable in t and ω) with respect to $\{\mathcal{F}_t\}_{t\geq t_0}$, $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, $D \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, and

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s))\mathrm{d}s + \int_{t_0}^{t} D(x(s))\mathrm{d}w(s) \ \ \text{a.s.}, \quad t \in [t_0, \tau], \tag{3.2}$$

where the integrals in (3.2) are Itô integrals.

Note that for each fixed $t \geq t_0$, the random variable $\omega \mapsto x(t,\omega)$ assigns a vector $x(\omega)$ to every outcome $\omega \in \Omega$ of an experiment, and for each fixed $\omega \in \Omega$, the mapping $t \mapsto x(t,\omega)$ is the *sample path* of the stochastic process $x(t)$, $t \geq t_0$. A pathwise solution $t \mapsto x(t)$ of (3.1) in $(\Omega, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P}^{x_0})$ is said to be *right maximally* defined if x cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal pathwise solutions to (3.1) $(\Omega, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P}^{x_0})$ exist on $[t_0, \infty)$, and hence, we assume (3.1) is *forward complete*. Sufficient conditions for forward completeness or *global solutions* of (3.1) are given in [5, Corol. 6.3.5].

Furthermore, we assume that $f : \mathcal{D} \to \mathbb{R}^n$ and $D : \mathcal{D} \to \mathbb{R}^{n\times d}$ satisfy the uniform Lipschitz continuity condition

$$\|f(x) - f(y)\| + \|D(x) - D(y)\|_{\rm F} \leq L\|x - y\|, \quad x, y \in \mathcal{D}, \tag{3.3}$$

and the growth restriction condition

$$\|f(x)\|^2 + \|D(x)\|_{\rm F}^2 \leq L^2(1 + \|x\|^2), \quad x \in \mathcal{D}, \tag{3.4}$$

for some Lipschitz constant $L > 0$, and hence, since $x(t_0) \in \mathcal{H}_n^{\mathcal{D}}$ and $x(t_0)$ is independent of $(w(t) - w(t_0)), t \geq t_0$, it follows that there exists a unique solution $x \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$

to (3.1) in the following sense. For every $x \in \mathcal{H}_n^{\mathcal{D}} \setminus \{0\}$ there exists $\tau_x > 0$ such that if $x_1 : [t_0, \tau_1] \times \Omega \to \mathcal{D}$ and $x_2 : [t_0, \tau_2] \times \Omega \to \mathcal{D}$ are two solutions of (3.1); that is, if $x_1, x_2 \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ with continuous sample paths almost surely solve (3.1), then $\tau_x \leq \min\{\tau_1, \tau_2\}$ and $\mathbb{P}\big(x_1(t) = x_2(t),\ t_0 \leq t \leq \tau_x\big) = 1$.

A weaker sufficient condition for the existence of a unique solution to (3.1) using a notion of (finite or infinite) escape time under the local Lipschitz continuity condition (3.3) without the growth condition (3.4) is given in [153]. Moreover, the unique solution determines a $\mathbb{R}^n$-valued, time-homogeneous Feller continuous Markov process $x(\cdot)$, and hence, its stationary Feller transition probability function is given by $\big($[70, Thm. 3.4], [5, Thm. 9.2.8]$\big)$

$$\mathbb{P}(x(t) \in \mathcal{B} | x(t_0) \stackrel{\text{a.s.}}{=} x_0) = \mathbb{P}(t - t_0, x_0, 0, \mathcal{B}), \quad x_0 \in \mathbb{R}^n, \tag{3.5}$$

for all $t \geq t_0$ and all Borel subsets $\mathcal{B}$ of $\mathbb{R}^n$, where $\mathbb{P}(s, x, t, \mathcal{B}), t \geq s$, denotes the probability of transition of the point $x \in \mathbb{R}^n$ at time instant s into the set $\mathcal{B} \subset \mathbb{R}^n$ at time instant t. Finally, recall that every continuous process with Feller transition probability function is also a strong Markov process [70, p.101].

Definition 3.1 [94, Def. 7.7]. Let $x(\cdot)$ be a time-homogeneous Markov process in $\mathcal{H}_n^{\mathcal{D}}$ and let $V : \mathcal{D} \to \mathbb{R}$. Then the *infinitesimal generator* $\mathcal{L}$ of $x(t)$, $t \geq 0$, with $x(0) \stackrel{\text{a.s.}}{=} x_0$, is defined by

$$\mathcal{L}V(x_0) \triangleq \lim_{t \to 0^+} \frac{\mathbb{E}^{x_0}[V(x(t))] - V(x_0)}{t}, \quad x_0 \in \mathcal{D}. \tag{3.6}$$

If $V \in \mathrm{C}^2$ and has a compact support, and $x(t)$, $t \geq t_0$, satisfies (3.1), then the limit in (3.6) exists for all $x \in \mathcal{D}$ and the infinitesimal generator $\mathcal{L}$ of $x(t)$, $t \geq t_0$, can be characterized by the system *drift* and *diffusion* functions $f(x)$ and $D(x)$ defining the stochastic dynamical system (3.1) and is given by ([94, Thm. 7.9])

$$\mathcal{L}V(x) \triangleq \frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2} \mathrm{tr}\ D^{\mathrm{T}}(x) \frac{\partial^2 V(x)}{\partial x^2} D(x), \quad x \in \mathcal{D}. \tag{3.7}$$

The following definition introduces the notions of Lyapunov and asymptotic stability in probability.

Definition 3.2 [70]. *i*) The equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\mathrm{e}}$ to (3.1) is *Lyapunov stable in probability* if, for every $\varepsilon > 0$ and $\rho \in (0,1)$, there exist $\delta = \delta(\rho, \varepsilon) > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(x_{\mathrm{e}})$,

$$\mathbb{P}^{x_0}\left(\sup_{t \geq t_0} \|x(t) - x_{\mathrm{e}}\| > \varepsilon\right) \leq \rho. \tag{3.8}$$

ii) The equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\mathrm{e}}$ to (3.1) is *asymptotically stable in probability* if it is Lyapunov stable in probability and there exists $\delta > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(x_{\mathrm{e}})$,

$$\lim_{x_0 \to x_{\mathrm{e}}} \mathbb{P}^{x_0}\left(\lim_{t \to \infty} \|x(t) - x_{\mathrm{e}}\| = 0\right) = 1. \tag{3.9}$$

Equivalently, the equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\mathrm{e}}$ to (3.1) is *locally asymptotically stable in probability* if it is Lyapunov stable in probability and, for every $\rho \in (0,1)$, there exist $\delta = \delta(\rho) > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(x_{\mathrm{e}})$,

$$\mathbb{P}^{x_0}\left(\lim_{t \to \infty} \|x(t) - x_{\mathrm{e}}\| = 0\right) \geq 1 - \rho. \tag{3.10}$$

iii) The equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\mathrm{e}}$ to (3.1) is *globally asymptotically stable in probability* if it is Lyapunov stable in probability and, for all $x_0 \in \mathbb{R}^n$,

$$\mathbb{P}^{x_0}\left(\lim_{t \to \infty} \|x(t) - x_{\mathrm{e}}\| = 0\right) = 1. \tag{3.11}$$

Next, we provide sufficient conditions for local and global asymptotic stability in probability for the nonlinear stochastic dynamical system (3.1).

Theorem 3.1 [70, Thm. 5.3, Corol. 5.1, and Thm. 5.11]. Consider the nonlinear stochastic dynamical system (3.1) and assume that there exists a two-times continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ such that

$$V(x_{\mathrm{e}}) = 0, \tag{3.12}$$

$$V(x) > 0, \qquad x \in \mathcal{D}, \qquad x \neq x_{\mathrm{e}}, \tag{3.13}$$

$$\frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2} \mathrm{tr}\, D^{\mathrm{T}}(x) \frac{\partial^2 V(x)}{\partial x^2} D(x) \leq 0, \qquad x \in \mathcal{D}. \tag{3.14}$$

Then the equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\text{e}}$ to (3.1) is Lyapunov stable in probability. If, in addition,

$$\frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2}\text{tr } D^{\text{T}}(x)\frac{\partial^2 V(x)}{\partial x^2} D(x) < 0, \quad x \in \mathcal{D}, \quad x \neq x_{\text{e}}, \tag{3.15}$$

then the equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\text{e}}$ to (3.1) is asymptotically stable in probability. Moreover, if $\mathcal{D} = \mathbb{R}^n$ and $V(\cdot)$ is radially unbounded, then the equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\text{e}}$ to (3.1) is globally asymptotically stable in probability.

The following definition introduces the notions of boundedness and uniform ultimate boundednesss for stochastic dynamical systems.

Definition 3.3 [141], [160]. The pathwise trajectory $x(t) \in \mathcal{H}_n^D$, $t \geq 0$, of (3.1) in $(\Omega, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P}^{x_0})$ is *bounded in probability* if $\lim_{c\to\infty} \sup_{t\geq 0} \mathbb{P}\{\|x(t)\| > c\} = 0$. Furthermore, $x(t) \in \mathcal{H}_n^D$, $t \geq 0$, is *uniformly ultimately bounded in the pth moment* if, for every compact subset $\mathcal{D}_{\text{c}} \subset \mathbb{R}^n$ and all $x(0) \overset{\text{a.s.}}{=} x_0 \in \mathcal{D}_{\text{c}}$, there exist $\varepsilon > 0$ and a finite time $T = T(\varepsilon, x_0)$ such that $\mathbb{E}^{x_0}[\|x(t)\|^p] < \varepsilon$ for all $t > 0 + T$. If, in addition, $p = 2$, then we say that $x(t)$, $t \geq 0$, is *uniformly ultimately bounded in a mean-square sense.*

Lemma 3.1 [31]. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (3.1). If there exist a two-times continuously differentiable function $V : \mathbb{R}^n \to \overline{\mathbb{R}}_+$, positive constants $\beta_1 > 0$ and $\beta_2 > 0$, and class $\mathcal{K}_\infty$ functions $\alpha_1 : [0, \infty) \to [0, \infty)$ and $\alpha_2 : [0, \infty) \to [0, \infty)$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad x \in \mathbb{R}^n, \tag{3.16}$$

$$\mathcal{L}V(x) \leq -\beta_1 V(x) + \beta_2, \quad x \in \mathbb{R}^n, \tag{3.17}$$

then

$$\mathbb{E}^{x_0}[V(x(t))] \leq V(x(0))e^{-\beta_1 t} + \frac{\beta_2}{\beta_1}, \quad t \geq 0. \tag{3.18}$$

3.3. Problem Formulation

In this section, we consider stochastic dynamical systems $\mathcal{G}$ of the form

$$\mathrm{d}x(t) = [Ax(t) + Bu(t)]\mathrm{d}t + x(t)g^{\mathrm{T}}\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{3.19}$$

where $x(t) \in \mathcal{H}_n$, $t \geq 0$, is the random state vector, $u(t) \in \mathcal{H}_m$, $t \geq 0$, is the uncompromised control input, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are known system matrices, $w(\cdot)$ is a d-dimensional independent standard Wiener process (i.e., Brownian motion) defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and $g \in \mathbb{R}^d$. Furthermore, we assume that the pair (A, B) is controllable and the control input $u(\cdot)$ satisfies sufficient regularity conditions such that (3.19) has a unique solution forward in time. Specifically, we assume that the control process $u(\cdot)$ in (3.19) is restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that $u(t) \in \mathcal{H}_m$, $t \geq 0$, and, for all $t \geq s$, $w(t) - w(s)$ is independent of $u(\tau)$, $w(\tau)$, $\tau \leq s$, and $x(0)$, and hence, $u(\cdot)$ is nonanticipative. In addition, we assume that $u(\cdot)$ takes values in a compact metrizable set, and hence, it follows from Theorem 2.2.4 of [4] that there exists a unique pathwise solution to (3.19) in $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}^{x_0})$.

We assume that the compromised system state is given by

$$\tilde{x}(t) = x(t) + \delta_{\mathrm{s}}(x(t)), \quad t \geq 0, \tag{3.20}$$

and is available for feedback, where $\tilde{x}(t) \in \mathcal{H}_n$, $t \geq 0$, and $\delta_{\mathrm{s}} : \mathcal{H}_n \to \mathcal{H}_n$ captures sensor attacks. Specifically, if $\delta_{\mathrm{s}}(\cdot)$ is nonzero, then the uncompromised random state vector $x(t)$, $t \geq 0$, is corrupted by a faulty (or malicious) signal $\delta_{\mathrm{s}}(\cdot)$. Alternatively, if $\mathbb{P}^{x_0}(\delta_{\mathrm{s}}(x(t)) = 0) = 1$, then $\tilde{x}(t) \overset{\text{a.s.}}{=} x(t)$, $t \geq 0$, and the uncompromised random state vector is available for feedback.

Here, we assume that the sensor attack in (3.20) is parameterized as $\delta_{\mathrm{s}}(x) = qx$, where $q \in \mathbb{R}$ is the sensor uncertainty, and hence, by (3.20) we obtain $\tilde{x} = (1 + q)x$. Thus, we assume that $q > -1$ in order to construct a feasible corrective signal $v(t)$, $t \geq 0$, since $q = -1$ results in $\tilde{x}(t) \overset{\text{a.s.}}{=} 0$, and hence, it is not possible to construct $v(t)$, $t \geq 0$, to asymptotically recover the ideal system performance.

Furthermore, we assume that the control input is also compromised and is given by

$$\tilde{u}(t) = u(t) + \delta_{\mathrm{a}}(\tilde{x}(t)), \quad t \geq 0, \tag{3.21}$$

where $\tilde{u}(t) \in \mathcal{H}_m$, $t \geq 0$, denotes the compromised control command signal and $\delta_{\mathrm{a}} : \mathcal{H}_n \to \mathcal{H}_m$ captures actuator attacks. In particular, if $\delta_{\mathrm{a}}(\cdot)$ is nonzero, then the uncompromised control signal $u(t)$, $t \geq 0$, is corrupted with a faulty (or malicious) signal $\delta_{\mathrm{a}}(\cdot)$. Alternatively, if $\mathbb{P}^{x_0}(\delta_{\mathrm{a}}(x(t)) = 0) = 1$, then $\tilde{u}(t) \overset{\text{a.s.}}{=} u(t)$, $t \geq 0$, and the control signal is uncompromised.

Here, we assume that the actuator attack in (3.21) can be parameterized as

$$\delta_{\mathrm{a}}(\tilde{x}) = W^{\mathrm{T}}\varphi(\tilde{x}) + \sigma(\tilde{x}), \tag{3.22}$$

where $W \in \mathbb{R}^{p \times m}$ is an *unknown* weighting matrix, $\varphi(\cdot) \in \mathbb{R}^p$ is a known nonlinear function, and $\sigma(\tilde{x}) \in \mathbb{R}^m$ is *unknown* and assumed to be bounded, that is, $\|\sigma(\tilde{x})\| \leq \bar{\sigma}$, $\tilde{x} \in \mathbb{R}^n$, and $\bar{\sigma} > 0$ is *unknown*. Note that assuming that $\|\sigma(\tilde{x})\| \leq \bar{\sigma}$, $\tilde{x} \in \mathbb{R}^n$, is without loss of generality since a worst-case actuator attack will lead to actuator amplitude saturation in practice. Alternatively, we can assume that $\sigma(\tilde{x})$, $\tilde{x} \in \mathbb{R}^n$, satisfies a Lipschitz condition.

The sensor attack model captures a multiplicative attack, wherein the attacker can corrupt the sensor measurements in a relative sense. For example, under this multiplicative attack model a malicious attack on a vehicle speed sensor will display a fraction of the vehicle's speed resulting in an unintentional increase in the vehicle's regulated velocity. Alternatively, the actuator attack model is an additive state dependent signal that accounts for a parameterization of the system attack modes as well as any residual signals.

Note that the compromised controlled system is given by

$$\mathrm{d}x(t) = [Ax(t) + B\tilde{u}(t)]\mathrm{d}t + x(t)g^{\mathrm{T}}\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0. \tag{3.23}$$

For $\delta_{\mathrm{s}}(x(t)) \neq 0$, $t \geq 0$, and $\delta_{\mathrm{a}}(x(t)) \neq 0$, $t \geq 0$, a.s. our objective is to design a feedback controller $\mathcal{G}_{\mathrm{c}}$ of the form

$$u(t) = K\tilde{x}(t) + v(t), \quad t \geq 0, \tag{3.24}$$

where $v(t) \in \mathcal{H}_m$, $t \geq 0$, is a corrective signal that suppresses or counteracts the effect of the state-dependent sensor and actuator attacks $\delta_{\mathrm{s}}(x(t))$, $t \geq 0$, and $\delta_{\mathrm{a}}(x(t))$, $t \geq 0$, to approximately recover the ideal system performance achieved when the uncompromised state vector is available for feedback and control signal is uncompromised.

3.4. Adaptive Control for State-Dependent Sensor and Actuator Attacks

In this section, we design the corrective signal $v(t)$, $t \geq 0$, in (3.24) to achieve adaptive ultimate boundedness in the presence of state-dependent sensor and actuator attacks. First, note that (A, B) is controllable if and only if $(A + \frac{1}{2}\|g\|^2 I_n, B)$ is controllable. Hence, there exists a feedback gain matrix $K \in \mathbb{R}^{n\times m}$ such that $A_{\mathrm{r}} + \frac{1}{2}\|g\|^2 I_n$ is Hurwitz, where $A_{\mathrm{r}} \triangleq A + BK$. In this case, it follows from converse Lyapunov theory [51] that for every positive definite matrix $R \in \mathbb{R}^{n\times n}$, there exists a unique positive definite matrix $P \in \mathbb{R}^{n\times n}$ satisfying

$$0 = \left(A_{\mathrm{r}} + \frac{1}{2}\|g\|^2 I_n\right)^{\mathrm{T}} P + P\left(A_{\mathrm{r}} + \frac{1}{2}\|g\|^2 I_n\right) + R. \tag{3.25}$$

To achieve ultimate boundedness in the face of state-dependent sensor and actuator attacks, we use the corrective signal given by

$$v(t) = -\hat{\mu}(t)K\tilde{x}(t) - \hat{W}^{\mathrm{T}}(t)\varphi(\tilde{x}(t)) - \hat{\sigma}(t)\mathrm{sgn}_{\mathrm{v}}(B^{\mathrm{T}}P\tilde{x}(t)), \quad t \geq 0, \tag{3.26}$$

where, for $y \in \mathbb{R}^m$, $\mathrm{sgn}_{\mathrm{v}}(y) \triangleq [\mathrm{sgn}(y_1), \ldots, \mathrm{sgn}(y_m)]^{\mathrm{T}}$, $\mathrm{sgn}(\alpha) \triangleq \frac{\alpha}{|\alpha|}$, $\alpha \neq 0$, and $\mathrm{sgn}(0) \triangleq 0$, with update laws

$$\mathrm{d}\hat{\mu}(t) = [\gamma\tilde{x}^{\mathrm{T}}(t)PBK\tilde{x}(t) - \xi_1\hat{\mu}(t)]\mathrm{d}t, \quad \hat{\mu}(0) \overset{\mathrm{a.s.}}{=} \hat{\mu}_0, \quad t \geq 0, \tag{3.27}$$

$$\mathrm{d}\hat{W}(t) = [\eta\varphi(\tilde{x}(t))\tilde{x}^{\mathrm{T}}(t)PB - \xi_2\hat{W}(t)]\mathrm{d}t, \quad \hat{W}(0) \overset{\text{a.s.}}{=} \hat{W}_0, \tag{3.28}$$

$$\mathrm{d}\hat{\sigma}(t) = [\nu\|\tilde{x}^{\mathrm{T}}(t)PB\|_1 - \xi_3\hat{\sigma}(t)]\mathrm{d}t, \quad \hat{\sigma}(0) \overset{\text{a.s.}}{=} \hat{\sigma}_0, \tag{3.29}$$

where $\hat{\mu}(t) \in \mathcal{H}$, $t \geq 0$, is the estimate of $\mu \triangleq q(1+q)^{-1} \in \mathbb{R}$ that depends on the sensor uncertainty q, $\hat{W}(t) \in \mathcal{H}_{p\times m}$, $t \geq 0$, is the estimate of the parametric uncertainty W, $\hat{\sigma}(t) \in \mathcal{H}$, $t \geq 0$, is the estimate of the unknown bound $\bar{\sigma}$, and $\gamma \in \mathbb{R}$, $\eta \in \mathbb{R}$, $\nu \in \mathbb{R}$, $\xi_1 \in \mathbb{R}$, $\xi_2 \in \mathbb{R}$, and $\xi_3 \in \mathbb{R}$ are positive design gains.

Next, using (3.21) and (3.22), (3.23) can be equivalently written as

$$\mathrm{d}x(t) = \Big(Ax(t) + B\Big[u(t) + W^{\mathrm{T}}\varphi(\tilde{x}(t)) + \sigma(\tilde{x}(t))\Big]\Big)\,\mathrm{d}t + x(t)g^{\mathrm{T}}\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0. \tag{3.30}$$

Now, define $\tilde{\mu}(t) \triangleq \mu - \hat{\mu}(t)$, $t \geq 0$, $\tilde{W}(t) \triangleq W - \hat{W}(t)$, $t \geq 0$, $\tilde{\sigma}(t) \triangleq \bar{\sigma} - \hat{\sigma}(t)$, $t \geq 0$, and $\lambda \triangleq \big(1+q\big)^{-1}$. Since $q > -1$, note that μ and λ are well-defined and $\lambda > 0$. Next, using $qx = \mu\tilde{x}$, (3.20), (3.24), and (3.26), it follows from (3.30) that

$$\begin{aligned}
\mathrm{d}x(t) &= \Big(Ax(t) + BK\tilde{x}(t) + Bv(t) + B\Big[W^{\mathrm{T}}\varphi(\tilde{x}(t)) + \sigma(\tilde{x}(t))\Big]\Big)\,\mathrm{d}t + x(t)g^{\mathrm{T}}\mathrm{d}w(t) \\
&= \Big(Ax(t) + BKx(t) + BKqx(t) + Bv(t) + B\Big[W^{\mathrm{T}}\varphi(\tilde{x}(t)) + \sigma(\tilde{x}(t))\Big]\Big)\,\mathrm{d}t \\
&\quad + x(t)g^{\mathrm{T}}\mathrm{d}w(t) \\
&= \Big(A_{\mathrm{r}}x(t) + BK\mu\tilde{x}(t) - \hat{\mu}(t)BK\tilde{x}(t) - B\hat{W}^{\mathrm{T}}(t)\varphi(\tilde{x}(t)) \\
&\quad - B\hat{\sigma}(t)\mathrm{sgn}_{\mathrm{v}}(B^{\mathrm{T}}P\tilde{x}(t)) + B\Big[W^{\mathrm{T}}\varphi(\tilde{x}(t)) + \sigma(\tilde{x}(t))\Big]\Big)\,\mathrm{d}t + x(t)g^{\mathrm{T}}\mathrm{d}w(t) \\
&= (A_{\mathrm{r}}x(t) + \tilde{\mu}(t)BK\tilde{x}(t) + B\tilde{W}^{\mathrm{T}}(t)\varphi(\tilde{x}(t)) + B(\sigma(\tilde{x}(t)) \\
&\quad - \hat{\sigma}(t)\mathrm{sgn}_{\mathrm{v}}(B^{\mathrm{T}}P\tilde{x}(t))))\mathrm{d}t + x(t)g^{\mathrm{T}}\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0.
\end{aligned} \tag{3.31}$$

Theorem 3.2. Consider the stochastic dynamical system $\mathcal{G}$ given by (3.23) with sensor and actuator attacks given by (3.20) and (3.21), respectively. Then, with the controller $\mathcal{G}_{\mathrm{c}}$ given by (3.24), the corrective signal $v(t)$, $t \geq 0$, given by (3.26), and adaptive laws given by (3.27)–(3.29), the closed-loop system given by (3.27)–(3.29) and (3.31) satisfies

$$\limsup_{t\to\infty} \mathbb{E}^{x_0}[\|x(t)\|^2] \leq \frac{c_0}{c_1\lambda_{\min}(P)}, \tag{3.32}$$

where $c_0 \triangleq \gamma^{-1}\xi_1\lambda\mu^2 + \eta^{-1}\xi_2\lambda\mathrm{tr}(W^{\mathrm{T}}W) + \nu^{-1}\xi_3\lambda\bar{\sigma}^2$ and $c_1 \triangleq \min\left\{\frac{\lambda_{\min}(R)}{\lambda_{\max}(P)}, \xi_1, \xi_2, \xi_3\right\}$. In addition, the adaptive estimates $\hat{\mu}(t)$, $t \geq 0$, $\hat{W}(t)$, $t \geq 0$, and $\hat{\sigma}(t)$, $t \geq 0$, are ultimately uniformly bounded in a mean-square sense.

Proof. To show ultimate boundedness of the closed-loop system, consider the Lyapunov-like function given by

$$V\big(x, \tilde{\mu}, \tilde{W}, \tilde{\sigma}\big) = x^{\mathrm{T}}Px + \frac{\lambda}{\gamma}\tilde{\mu}^2 + \frac{\lambda}{\eta}\mathrm{tr}(\tilde{W}^{\mathrm{T}}\tilde{W}) + \frac{\lambda}{\nu}\tilde{\sigma}^2, \tag{3.33}$$

where P satisfies (3.25). Now, the corresponding infinitesimal generator $\mathcal{L}V\big(x, \tilde{\mu}, \tilde{W}, \tilde{\sigma}\big)$ satisfies

$$\begin{aligned}\mathcal{L}V\big(x, \tilde{\mu}, \tilde{W}, \tilde{\sigma}\big) &= x^{\mathrm{T}}PA_{\mathrm{r}}x + x^{\mathrm{T}}A_{\mathrm{r}}^{\mathrm{T}}Px + 2\tilde{\mu}x^{\mathrm{T}}PBK\tilde{x} + 2x^{\mathrm{T}}PB\tilde{W}^{\mathrm{T}}\varphi(\tilde{x})\\
&\quad + 2x^{\mathrm{T}}PB(\sigma(\tilde{x}) - \hat{\sigma}\mathrm{sgn}_{\mathrm{v}}(B^{\mathrm{T}}P\tilde{x})) + \mathrm{tr}(gx^{\mathrm{T}}Pxg^{\mathrm{T}})\\
&\quad + 2\frac{\lambda}{\gamma}\tilde{\mu}[-\gamma\tilde{x}^{\mathrm{T}}PBK\tilde{x} + \xi_1\hat{\mu}] + 2\frac{\lambda}{\eta}\mathrm{tr}[\tilde{W}^{\mathrm{T}}(-\eta\varphi(\tilde{x})\tilde{x}^{\mathrm{T}}PB + \xi_2\hat{W})]\\
&\quad + 2\frac{\lambda}{\nu}\tilde{\sigma}[-\nu\|\tilde{x}^{\mathrm{T}}BP\|_1 + \xi_3\hat{\sigma}], \quad (x, \tilde{\mu}, \tilde{W}, \tilde{\sigma}) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{p\times m} \times \mathbb{R}.\end{aligned} \tag{3.34}$$

Noting that $\mathrm{tr}(gx^{\mathrm{T}}Pxg^{\mathrm{T}}) = \mathrm{tr}(x^{\mathrm{T}}Pxg^{\mathrm{T}}g) = x^{\mathrm{T}}Pxg^{\mathrm{T}}g = x^{\mathrm{T}}Px\|g\|^2$ and $2x^{\mathrm{T}}PB\tilde{W}^{\mathrm{T}}\varphi(\tilde{x}) = 2\mathrm{tr}[\tilde{W}^{\mathrm{T}}\varphi(\tilde{x})x^{\mathrm{T}}PB]$, it follows from (3.34) that

$$\begin{aligned}\mathcal{L}V\big(x, \tilde{\mu}, \tilde{W}, \tilde{\sigma}\big) &= x^{\mathrm{T}}\left[\left(A_{\mathrm{r}} + \frac{1}{2}\|g\|^2I_n\right)^{\mathrm{T}}P + P\left(A_{\mathrm{r}} + \frac{1}{2}\|g\|^2I_n\right)\right]x + 2\tilde{\mu}x^{\mathrm{T}}PBK\tilde{x}\\
&\quad + 2\frac{\lambda}{\gamma}\tilde{\mu}[-\gamma\tilde{x}^{\mathrm{T}}PBK\tilde{x} + \xi_1\hat{\mu}] + 2\mathrm{tr}[\tilde{W}^{\mathrm{T}}\varphi(\tilde{x})x^{\mathrm{T}}PB]\\
&\quad + 2\frac{\lambda}{\eta}\mathrm{tr}[\tilde{W}^{\mathrm{T}}(-\eta\varphi(\tilde{x})\tilde{x}^{\mathrm{T}}PB + \xi_2\hat{W})]\\
&\quad + 2x^{\mathrm{T}}PB(\sigma(\tilde{x}) - \hat{\sigma}\mathrm{sgn}_{\mathrm{v}}(B^{\mathrm{T}}P\tilde{x})) + 2\frac{\lambda}{\nu}\tilde{\sigma}[-\nu\|\tilde{x}^{\mathrm{T}}BP\|_1 + \xi_3\hat{\sigma}],\\
&\qquad (x, \tilde{\mu}, \tilde{W}, \tilde{\sigma}) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{p\times m} \times \mathbb{R}.\end{aligned} \tag{3.35}$$

Next, using $\tilde{x}^{\mathrm{T}}PB\sigma(\tilde{x}) \leq \|\tilde{x}^{\mathrm{T}}PB\|\bar{\sigma} \leq \|\tilde{x}^{\mathrm{T}}PB\|_1\bar{\sigma}$ and $\tilde{x}^{\mathrm{T}}PB\mathrm{sgn}_{\mathrm{v}}(B^{\mathrm{T}}P\tilde{x}) = \|\tilde{x}^{\mathrm{T}}PB\|_1$, it follows that

$$2x^{\mathrm{T}}PB[\sigma(\tilde{x}) - \hat{\sigma}\mathrm{sgn}_{\mathrm{v}}(B^{\mathrm{T}}P\tilde{x})] \leq 2\lambda\|\tilde{x}^{\mathrm{T}}PB\|_1\overline{\sigma} - 2\lambda\|\tilde{x}^{\mathrm{T}}PB\|_1\hat{\sigma}$$

$$= 2\lambda\|\tilde{x}^{\mathrm{T}}PB\|_1\tilde{\sigma}, \quad x \in \mathbb{R}^n. \tag{3.36}$$

Now, using (3.25), it follows from (3.35) that

$$\begin{aligned}
\mathcal{L}V\big(x,\tilde{\mu},\tilde{W},\tilde{\sigma}\big) \le{}& -x^{\mathrm{T}}Rx + 2\tilde{\mu}\lambda\tilde{x}^{\mathrm{T}}PBK\tilde{x} + 2\frac{\lambda}{\gamma}\tilde{\mu}[-\gamma\tilde{x}^{\mathrm{T}}PBK\tilde{x} + \xi_1\hat{\mu}] \\
&+ 2\mathrm{tr}[\tilde{W}^{\mathrm{T}}\lambda\varphi(\tilde{x})\tilde{x}^{\mathrm{T}}PB] + 2\frac{\lambda}{\eta}\mathrm{tr}[\tilde{W}^{\mathrm{T}}(-\eta\varphi(\tilde{x})\tilde{x}^{\mathrm{T}}PB + \xi_2\hat{W})] \\
&+ 2\|\tilde{x}^{\mathrm{T}}BP\|_1\tilde{\sigma}\lambda + 2\frac{\lambda}{\nu}\tilde{\sigma}[-\nu\|\tilde{x}^{\mathrm{T}}BP\|_1 + \xi_3\hat{\sigma}] \\
\le{}& -x^{\mathrm{T}}Rx + 2\frac{\lambda}{\gamma}\xi_1\tilde{\mu}\hat{\mu} + 2\frac{\lambda}{\eta}\xi_2\mathrm{tr}(\tilde{W}^{\mathrm{T}}\hat{W}) \\
&+ 2\frac{\lambda}{\nu}\xi_3\tilde{\sigma}\hat{\sigma}, \quad (x,\tilde{\mu},\tilde{W},\tilde{\sigma}) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{p\times m} \times \mathbb{R}.
\end{aligned} \tag{3.37}$$

Next, writing $2\frac{\lambda}{\gamma}\xi_1\tilde{\mu}\hat{\mu}$ and $2\frac{\lambda}{\eta}\xi_2\mathrm{tr}(\tilde{W}^{\mathrm{T}}\hat{W})$ in (3.37), respectively, as

$$\begin{aligned}
2\frac{\lambda}{\gamma}\xi_1\tilde{\mu}\hat{\mu} &= 2\frac{\lambda}{\gamma}\xi_1\tilde{\mu}(\mu - \tilde{\mu}) = 2\frac{\lambda}{\gamma}\xi_1\tilde{\mu}\mu - 2\frac{\lambda}{\gamma}\xi_1\tilde{\mu}^2 \\
&\le \frac{\lambda}{\gamma}\xi_1\tilde{\mu}^2 + \frac{\lambda}{\gamma}\xi_1\mu^2 - 2\frac{\lambda}{\gamma}\xi_1\tilde{\mu}^2 \\
&= -\frac{\lambda}{\gamma}\xi_1\tilde{\mu}^2 + \frac{\lambda}{\gamma}\xi_1\mu^2,
\end{aligned} \tag{3.38}$$

where $\frac{\lambda}{\gamma}\xi_1\mu^2$ is a constant, and

$$2\frac{\lambda}{\eta}\xi_2\mathrm{tr}(\tilde{W}^{\mathrm{T}}\hat{W}) \le -\frac{\lambda}{\eta}\xi_2\mathrm{tr}(\tilde{W}^{\mathrm{T}}\tilde{W}) + \frac{\lambda}{\eta}\xi_2\mathrm{tr}(W^{\mathrm{T}}W), \tag{3.39}$$

$$2\frac{\lambda}{\nu}\xi_3\tilde{\sigma}\hat{\sigma} \le -\frac{\lambda}{\nu}\xi_3\tilde{\sigma}^2 + \frac{\lambda}{\nu}\xi_3\bar{\sigma}^2, \tag{3.40}$$

it follows from (3.37)–(3.40) that

$$\begin{aligned}
\mathcal{L}V\big(x,\tilde{\mu},\tilde{W},\tilde{\sigma}\big) \le{}& -x^{\mathrm{T}}Rx - \gamma^{-1}\lambda\xi_1\tilde{\mu}^2 - \eta^{-1}\lambda\xi_2\mathrm{tr}(\tilde{W}^{\mathrm{T}}\tilde{W}) - \nu^{-1}\lambda\xi_3\tilde{\sigma}^2 + \gamma^{-1}\xi_1\lambda\mu^2 \\
&+ \eta^{-1}\xi_2\lambda\mathrm{tr}(W^{\mathrm{T}}W) + \nu^{-1}\xi_3\lambda\bar{\sigma}^2 \\
\le{}& -\frac{\lambda_{\min}(R)}{\lambda_{\max}(P)}x^{\mathrm{T}}Px - \gamma^{-1}\lambda\xi_1\tilde{\mu}^2 - \eta^{-1}\lambda\xi_2\mathrm{tr}(\tilde{W}^{\mathrm{T}}\tilde{W}) \\
&- \nu^{-1}\lambda\xi_3\tilde{\sigma}^2 + c_0 \\
\le{}& -c_1V\big(x,\tilde{\mu},\tilde{W},\tilde{\sigma}\big) + c_0, \quad (x,\tilde{\mu},\tilde{W},\tilde{\sigma}) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{p\times m} \times \mathbb{R},
\end{aligned} \tag{3.41}$$

where $c_0 \triangleq \frac{\lambda}{\gamma}\xi_1\mu^2 + \frac{\lambda}{\eta}\xi_2\mathrm{tr}(W^\mathrm{T}W) + \frac{\lambda}{\nu}\xi_3\bar{\sigma}^2$ and $c_1 \triangleq \min\left\{\frac{\lambda_{\min}(R)}{\lambda_{\max}(P)}, \xi_1, \xi_2, \xi_3\right\}$. Thus, using Lemma 3.1, it follows from (3.41) that

$$0 \leq \mathbb{E}^{x_0}[V\big(x(t), \tilde{\mu}(t), \tilde{W}(t), \tilde{\sigma}(t)\big)] \leq V\big(x(0), \tilde{\mu}(0), \tilde{W}(0), \tilde{\sigma}(0)\big)e^{-c_1 t} + \frac{c_0}{c_1}, \quad t \geq 0, \tag{3.42}$$

and hence, all the signals of the closed-loop system are uniformly ultimately bounded in probability in a mean-square sense. Furthermore, noting that

$$\limsup_{t\to\infty} \mathbb{E}^{x_0}[x^\mathrm{T}(t)Px(t)] \leq \limsup_{t\to\infty} \mathbb{E}^{x_0}[V\big(x(t), \tilde{\mu}(t), \tilde{W}(t), \tilde{\sigma}(t)\big)] \leq \frac{c_0}{c_1}, \tag{3.43}$$

it follows that

$$\limsup_{t\to\infty} \mathbb{E}^{x_0}[\|x(t)\|^2] \leq \frac{c_0}{c_1\lambda_{\min}(P)}, \tag{3.44}$$

which implies that the pathwise trajectory of the closed-loop system associated with the plant dynamics is uniformly ultimately bounded in a mean-square sense. □

Theorem 3.2 assumes that $q > -1$. This assumption implies that $\lambda > 0$. As long as the sign of λ is known, Theorem 3.2 can be used to address the case where $\lambda < 0$. The assumption $q > -1$ can be relaxed by utilizing tools from [73] that can allow λ to have any sign as long as $q \neq -1$ under the assumption that its sign is a priori known.

In arriving at Theorem 3.2 we assumed that $\|\sigma(\tilde{x})\| \leq \bar{\sigma}$, $\tilde{x} \in \mathbb{R}^n$, where $\bar{\sigma} > 0$ is unknown. Alternatively, we can assume that $\sigma(\tilde{x})$ satisfies the Lipschitz condition $\|\sigma(\tilde{x})\| \leq \bar{\sigma}\|\tilde{x}\|$, $\tilde{x} \in \mathbb{R}^n$, where $\bar{\sigma} > 0$ is an *unknown* Lipschitz constant. In this case, it can be shown that Theorem 3.1 holds with (3.26) replaced by

$$v(t) = -\hat{\mu}(t)K\tilde{x}(t) - \hat{W}^\mathrm{T}(t)\varphi(\tilde{x}(t)) - \hat{\sigma}(t)\|\tilde{x}(t)\|\mathrm{sgn_v}(B^\mathrm{T}P\tilde{x}(t)), \quad t \geq 0, \tag{3.45}$$

and with (3.29) replaced by

$$\mathrm{d}\hat{\sigma}(t) = [\nu\|\tilde{x}(t)\|\|\tilde{x}^\mathrm{T}(t)PB\|_1 - \xi_3\hat{\sigma}(t)]\mathrm{d}t, \quad \hat{\sigma}(0) \overset{\text{a.s.}}{=} \hat{\sigma}_0, \quad t \geq 0. \tag{3.46}$$

Note that the controller $u(t)$, $t \geq 0$, given by (3.24) is discontinuous because of the presence of the signum function $\mathrm{sgn_v}(\cdot)$ in the controller architecture. This discontinuity can lead to a chattering phenomenon, which is undesirable in practice. In order to reduce or eliminate the chattering effect, a smooth function can be implemented instead of the signum function [109]; that is, we replace $\mathrm{sgn_v}(\cdot)$ by $\mathrm{tanh_v}(\cdot)$, where, for $y \in \mathbb{R}^n$, $\mathrm{tanh_v}(y) \triangleq [\tanh(y_1), \ldots, \tanh(y_n)]^{\mathrm{T}}$. Note that ([109])

$$0 \le |\alpha| - \alpha \tanh\left(\frac{\alpha}{\varepsilon}\right) \le k_0\varepsilon, \qquad \alpha \in \mathbb{R}, \tag{3.47}$$

where $\varepsilon > 0$ is a design constant and k_0 satisfies $k_0 = e^{-(k_0+1)}$ with $k_0 = 0.2785$. Thus, we modify (3.26) as

$$v(t) = -\hat{\mu}(t)K\tilde{x}(t) - \hat{W}^{\mathrm{T}}(t)\varphi(\tilde{x}(t)) - \hat{\sigma}(t)\mathrm{tanh_v}\left(\frac{B^{\mathrm{T}}P\tilde{x}(t)\hat{\sigma}(t)}{\varepsilon}\right), \quad t \ge 0. \tag{3.48}$$

In this case, (3.36) becomes

$$\begin{aligned}
&2x^{\mathrm{T}}PB\left[\sigma(x) - \hat{\sigma}\mathrm{tanh_v}\left(\frac{B^{\mathrm{T}}P\tilde{x}\hat{\sigma}}{\varepsilon}\right)\right] \\
&\quad \le 2\|\tilde{x}^{\mathrm{T}}PB\|_1\bar{\sigma}\lambda - 2\tilde{x}^{\mathrm{T}}PB\hat{\sigma}\lambda\mathrm{tanh_v}\left(\frac{B^{\mathrm{T}}P\tilde{x}\hat{\sigma}}{\varepsilon}\right) \\
&\quad = 2\|\tilde{x}^{\mathrm{T}}PB\|_1(\hat{\sigma}+\tilde{\sigma})\lambda - 2\tilde{x}^{\mathrm{T}}PB\hat{\sigma}\lambda\mathrm{tanh_v}\left(\frac{B^{\mathrm{T}}P\tilde{x}\hat{\sigma}}{\varepsilon}\right) \\
&\quad \le 2\|\tilde{x}^{\mathrm{T}}PB\|_1(|\hat{\sigma}|+\tilde{\sigma})\lambda - 2\tilde{x}^{\mathrm{T}}PB\hat{\sigma}\lambda\mathrm{tanh_v}\left(\frac{B^{\mathrm{T}}P\tilde{x}\hat{\sigma}}{\varepsilon}\right) \\
&\quad = 2\|\tilde{x}^{\mathrm{T}}PB\|_1\tilde{\sigma}\lambda + \sum_{i=1}^{m} 2\lambda\left[|(\tilde{x}^{\mathrm{T}}PB)_i\hat{\sigma} - (\tilde{x}^{\mathrm{T}}PB)_i\hat{\sigma}\tanh\left(\frac{(\tilde{x}^{\mathrm{T}}PB)_i\hat{\sigma}}{\varepsilon}\right)\right] \\
&\quad \le 2\|\tilde{x}^{\mathrm{T}}PB\|_1\tilde{\sigma}\lambda + 2m\lambda k_0\varepsilon.
\end{aligned} \tag{3.49}$$

Now, it follows from (3.41) that

$$\begin{aligned}
\mathcal{L}V\big(x,\tilde{\mu},\tilde{W},\tilde{\sigma}\big) &\le -x^{\mathrm{T}}Rx - \gamma^{-1}\lambda\xi_1\tilde{\mu}^2 - \eta^{-1}\lambda\xi_2\mathrm{tr}(\tilde{W}^{\mathrm{T}}\tilde{W}) - \nu^{-1}\lambda\xi_3\tilde{\sigma}^2 + \gamma^{-1}\xi_1\lambda\mu^2 \\
&\quad + \eta^{-1}\xi_2\lambda\mathrm{tr}(W^{\mathrm{T}}W) + \nu^{-1}\xi_3\lambda\bar{\sigma}^2 + 2m\lambda k_0\varepsilon \\
&\le -\frac{\lambda_{\min}(R)}{\lambda_{\max}(P)}x^{\mathrm{T}}Px - \gamma^{-1}\lambda\xi_1\tilde{\mu}^2 - \eta^{-1}\lambda\xi_2\mathrm{tr}(\tilde{W}^{\mathrm{T}}\tilde{W}) - \nu^{-1}\lambda\xi_3\tilde{\sigma}^2 + c_2 \\
&\le -c_1V\big(x,\tilde{\mu},\tilde{W},\tilde{\sigma}\big) + c_2, \quad (x,\tilde{\mu},\tilde{W},\tilde{\sigma}) \in \mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^{p\times m}\times\mathbb{R},
\end{aligned} \tag{3.50}$$

where $c_2 \triangleq \gamma^{-1}\xi_1\lambda\mu^2 + \eta^{-1}\xi_2\lambda\mathrm{tr}(W^{\mathrm{T}}W) + \nu^{-1}\xi_3\lambda\bar{\sigma}^2 + 2m\lambda k_0\varepsilon = c_0 + 2m\lambda k_0\varepsilon$ and $c_1 = \min\left\{\frac{\lambda_{\min}(R)}{\lambda_{\max}(P)}, \xi_1, \xi_2, \xi_3\right\}$. Using similar arguments as in the proof of Theorem 3.1, it can be shown that

$$\limsup_{t\to\infty} \mathbb{E}^{x_0}[\|x(t)\|^2] \leq \frac{c_2}{c_1\lambda_{\min}(P)}, \tag{3.51}$$

which is identical to the result of Theorem 3.1 with the only difference being that c_0 is replaced by c_2.

3.5. Illustrative Numerical Example

To illustrate the key ideas presented in Section 3.4, we consider a dynamical system representing the lateral directional dynamics of an aircraft [73] given by

$$\begin{bmatrix} \mathrm{d}\beta(t) \\ \mathrm{d}p(t) \\ \mathrm{d}r(t) \end{bmatrix} = \begin{bmatrix} -0.025 & 0.104 & -0.994 \\ 574.7 & 0 & 0 \\ 16.20 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta(t)\mathrm{d}t \\ p(t)\mathrm{d}t \\ r(t)\mathrm{d}t \end{bmatrix} + \begin{bmatrix} 0.122 & -0.276 \\ -53.61 & 33.25 \\ 195.5 & -529.4 \end{bmatrix} \begin{bmatrix} \delta_{\mathrm{ail}}(t)\mathrm{d}t \\ \delta_{\mathrm{rud}}(t)\mathrm{d}t \end{bmatrix},$$
$$+\frac{1}{2}\begin{bmatrix} \beta(t) \\ p(t) \\ r(t) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{\mathrm{T}} \mathrm{d}w(t), \qquad \begin{bmatrix} \beta(0) \\ p(0) \\ r(0) \end{bmatrix} \overset{\text{a.s.}}{=} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \quad t \geq 0, \tag{3.52}$$

with a linear-quadratic optimal state feedback control gain given by

$$K = \begin{bmatrix} 2.1864 & 0.0699 & -0.0587 \\ -3.9959 & -0.0996 & 0.1428 \end{bmatrix}. \tag{3.53}$$

The state vector $x(t) \triangleq \left[\beta(t),\ p(t),\ r(t)\right]^{\mathrm{T}}$, $t \geq 0$, contains the sideslip angle in deg, the roll rate in deg/sec, and the yaw rate in deg/sec, respectively, and the control input $u(t) \triangleq \left[\delta_{\mathrm{ail}}(t),\ \delta_{\mathrm{rud}}(t)\right]^{\mathrm{T}}$, $t \geq 0$, contains the aileron command in deg and the rudder command in deg, respectively. Here, the state-dependent disturbance in (3.52) is used to capture perturbations in atmospheric drag [82]. Figure 3.5.1 shows a sample trajectory along with the standard deviation of the state trajectories $x(t)$, $t \geq 0$, of the nominal system versus time for 30 sample paths. The mean control profile is also plotted in Figure 3.5.1.

To illustrate the results of Theorem 3.1 with (3.26) replaced by (3.48) consider the state-dependent sensor and actuator attacks given by (3.20) and (3.21) respectively, with $w = -0.75$, $\delta_{\mathrm{a}}(x(t)) = [4,\ 3]^{\mathrm{T}} 4\sin(\tilde{\beta}(t))\cos(\tilde{p}(t))$, $t \geq 0$, $W = [4,\ 3]$, and $\varphi(\tilde{x}(t)) =$

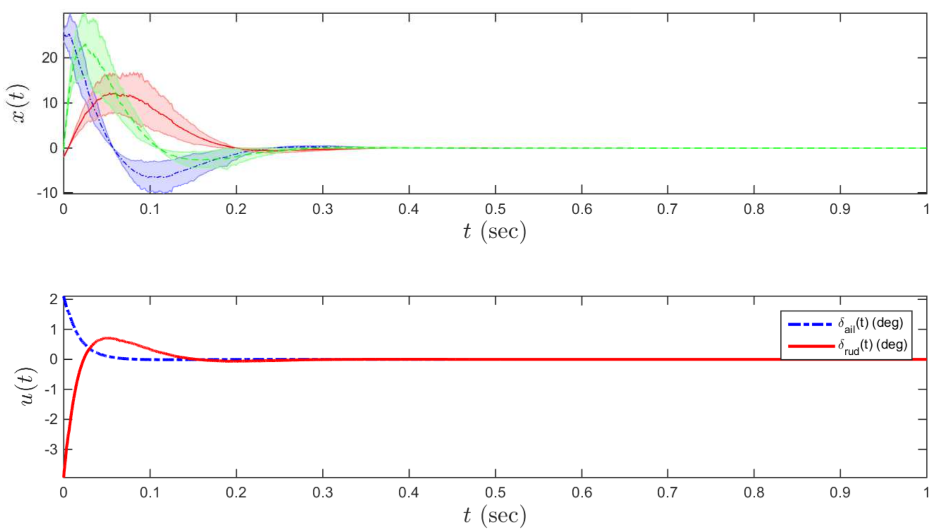


Figure 3.5.1: A sample trajectory along with the sample standard deviation of the closed-loop nominal system trajectories versus time; $\beta(t)$ in blue, $p(t)$ in red, and $r(t)$ in green. The control profile is plotted as the mean of the 30 sample runs.

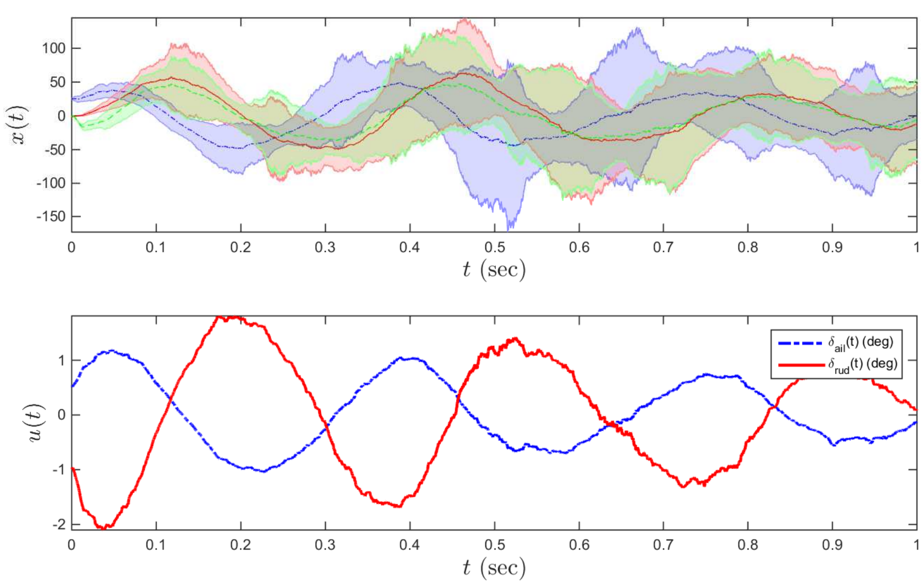


Figure 3.5.2: A sample trajectory along with the sample standard deviation of the closed-loop system trajectories versus time in the presence of state-dependent sensor and actuator attacks without any corrective action (i.e., $v(t) \equiv 0$ in (3.24)); $\beta(t)$ in blue, $p(t)$ in red, and $r(t)$ in green. The control profile is plotted as the mean of the 30 sample runs.

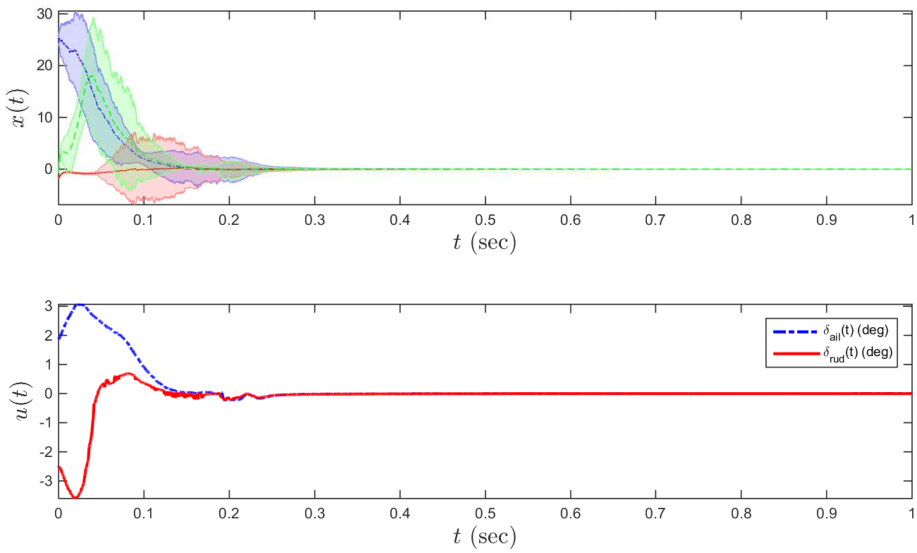


Figure 3.5.3: A sample trajectory along with the sample standard deviation of the closed-loop system trajectories versus time in the presence of state-dependent sensor and actuator attacks with the proposed corrective signal (3.48); $\beta(t)$ in blue, $p(t)$ in red, and $r(t)$ in green. The control profile is plotted as the mean of the 30 sample runs.

$4\sin(\tilde{\beta}(t))\cos(\tilde{p}(t))$, $t \geq 0$. Figure 3.5.2 shows a sample trajectory along with the standard deviation of the state trajectories $x(t)$, $t \geq 0$, of the system under attack without any corrective action for 30 sample paths. The mean control profile is also plotted in Figure 3.5.2.

To design the proposed adaptive control law and corrective signal given by (3.27)–(3.29), and (3.48), we set $\gamma = 0.8$, $\xi = 0.8$, $\eta = 0.8$, $\nu = 0.8$, $\xi_1 = 1$, $\xi_2 = 1$, $\xi_3 = 1$, $\epsilon = 0.8$, and $R = I_3$. In this case, P satisfying (3.25) is given by

$$P = \begin{bmatrix} 45.1270 & 1.3354 & -0.2058 \\ 1.3354 & 0.0695 & -0.0037 \\ -0.2058 & -0.0037 & 0.0079 \end{bmatrix}. \tag{3.54}$$

Alternatively, a more methodical selection using a convex optimization approach [38] can be used to select the design parameters. The system performance of the controller given by (3.24) with the proposed corrective signal is depicted in Figure 3.5.3. Specifically, Figure 3.5.3 shows a sample trajectory along with the standard deviation of the state trajectories

$x(t)$, $t \geq 0$, of the system versus time for 30 sample paths. The mean control profile is also plotted in Figure 3.5.3.

Chapter 4

Adaptive Control for Leader-Follower Stochastic Multiagent Systems with Sensor and Actuator Attacks

4.1. Introduction

In this chapter, we build on the results of Chapter 3 and multiagent systems theory to develop a new distributed adaptive control architecture that can foil malicious sensor and actuator attacks for networked systems in the face of exogenous stochastic disturbances. Specifically, for a class of linear multiagent systems with an undirected communication graph topology we develop a new structure of the neighborhood synchronization error for the distributed adaptive control protocol design of each follower to account for time-varying multiplicative sensor attacks on the leader state. In addition, the proposed framework accounts for time-varying multiplicative actuator attacks on the followers that do not have a communication link with the leader. Moreover, our framework addresses time-varying additive actuator attacks on all the follower agents in the network. The proposed controller guarantees uniform ultimate boundedness in probability of the state tracking error for each follower agent in a mean-square sense. Furthermore, to show the efficacy of our adaptive control architecture, we provide a numerical illustrative example involving the lateral directional dynamics of an aircraft group of agents subject to state-dependent atmospheric drag disturbances as well as sensor and actuator attacks.

Next, we extend our framework to develop a distributed robust adaptive control architecture that can foil malicious sensor and actuator attacks in the face of exogenous stochastic disturbances and follower agent model uncertainties. Specifically, for a class of linear multiagent uncertain systems with an undirected communication graph topology we develop a neighborhood synchronization error for the distributed robust adaptive control protocol design of each follower to account for actuator and sensor attacks on the leader state as well as all of the follower agents in the network. The proposed robust adaptive controller guarantees uniform ultimate boundedness in probability of the state tracking error for each follower agent in a mean-square sense. To show the efficacy of our adaptive control architecture, we provide a numerical illustrative example involving the lateral directional dynamics of an aircraft group of agents subject to state-dependent atmospheric drag disturbances, sensor and actuator attacks, and follower agent model uncertainties. Finally, to account for a more realistic setting, output feedback architectures are presented for leader-follower multiagent systems with stochastic disturbances and sensor and actuator attacks.

4.2. Notation and Definitions

Here, we recall some basic notation from graph theory [45]. Specifically, $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ denotes a weighted *directed graph* (or *digraph*) denoting the static network (or static graph) with the set of *nodes* (or *vertices*) $\mathcal{V} = \{0, 1, \ldots, N\}$ involving a finite nonempty set denoting the agents, the set of *edges* $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ involving a set of ordered pairs denoting the direction of information flow between agents, and a *weighted adjacency matrix* $\mathcal{A} \in \mathbb{R}^{(N+1)\times(N+1)}$ such that $\mathcal{A}_{(i,j)} = a_{ij} > 0$, $i, j \in \mathcal{V}$, if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$, otherwise. The edge $(j, i) \in \mathcal{E}$ denotes that agent i can obtain information from agent j, but not necessarily vice versa. Moreover, we assume that $a_{ii} = 0$ for all $i \in \mathcal{V}$.

Note that if the weights a_{ij}, $i, j \in \mathcal{V}$, are not relevant, then a_{ij} is set to 1 for all $(j, i) \in \mathcal{E}$. In this case, $\mathcal{A}$ is called a *normalized adjacency matrix*. Every edge $\ell \in \mathcal{E}$ corresponds to an ordered pair of vertices $(i,j) \in \mathcal{V} \times \mathcal{V}$, where i and j are the *initial* and *terminal* vertices of the edge ℓ. In this case, ℓ is *incident into* j and *incident out of* i. We say that $\mathfrak{G}$ is *strongly* (resp., *weakly*) *connected* if for every ordered pair of vertices (i,j), $i \neq j$, there exists a *directed* (resp., *undirected*) *path*, that is, a directed (resp., undirected) sequence of arcs leading from i to j. Furthermore, the *in-neighbors* and *out-neighbors* of node i are, respectively, defined as $\mathcal{N}_{\mathrm{in}}(i) \triangleq \{j \in \mathcal{V} : (j,i) \in \mathcal{E}\}$ and $\mathcal{N}_{\mathrm{out}}(i) \triangleq \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$.

A *graph* or *undirected graph* $\mathfrak{G}$ associated with the adjacency matrix $\mathcal{A} \in \mathbb{R}^{(N+1)\times(N+1)}$ is a directed graph for which the arc set is symmetric, that is, $\mathcal{A} = \mathcal{A}^{\mathrm{T}}$. Furthermore, in this case we say that $\mathfrak{G}$ is *connected* if for every ordered pair of vertices (i,j), $i \neq j$, there exists a *path*, that is, a sequence of arcs, leading from i to j. The *graph Laplacian* matrix denoted by $\mathfrak{L} \in \mathbb{R}^{(N+1)\times(N+1)}$ is defined as $\mathfrak{L}_{(i,i)} = \sum_{j=0,j\neq i}^{N} \mathcal{A}_{(i,j)}$ and $\mathfrak{L}_{(i,j)} = -\mathcal{A}_{(i,j)}$ for $i \neq j$. Finally, we denote the leader agent by index 0 and the follower agents by index $1,\ldots,N$, and assume the leader has no heighbours. In this case, the graph Laplacian matrix $\mathfrak{L}$ has the structure

$$\mathfrak{L} = \begin{bmatrix} 0 & 0_{1\times N} \\ \mathfrak{L}_2 & \mathfrak{L}_1 \end{bmatrix}, \tag{4.1}$$

where $\mathfrak{L}_1 \in \mathbb{R}^{N\times N}$ and $\mathfrak{L}_2 \in \mathbb{R}^{N\times 1}$. Furthermore, the set of nodes that do not have access to the leader information is denoted by $\mathcal{N}_{\mathrm{I}}$, whereas the set of nodes with access to the leader information is denoted by $\mathcal{N}_{\mathrm{II}}$. It is clear that $\mathcal{N}_{\mathrm{I}} \cap \mathcal{N}_{\mathrm{II}} = \emptyset$ and $\mathcal{N}_{\mathrm{I}} \cup \mathcal{N}_{\mathrm{II}} = \{1,\ldots,N\}$.

4.3. Adaptive Control for Multiagent Systems with Stochastic Disturbances, Actuator Attacks, and Compromised Leader State Measurements

Consider a leader-follower networked multiagent system consisting of N follower agents with the dynamics of agent $i \in \{1,\ldots,N\}$ given by

$$\mathrm{d}x_i(t) = [Ax_i(t) + Bu_i(t)]\mathrm{d}t + x_i(t)g^{\mathrm{T}}\mathrm{d}w(t), \quad x_i(0) \overset{\text{a.s.}}{=} x_{i0}, \quad t \geq 0, \tag{4.2}$$

where, for $t \geq 0$ and $i \in \{1,\ldots,N\}$, $x_i(t) \in \mathcal{H}_n$ is the state of the ith follower agent, $u_i(t) \in \mathcal{H}_m$ is the uncorrupted control input to the ith follower agent, $A \in \mathbb{R}^{n\times n}$ and

$B \in \mathbb{R}^{n\times m}$ are system matrices, $w(\cdot)$ is a d-dimensional independent standard Wiener process (i.e., Brownian motion) defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and $g \in \mathbb{R}^d$. Furthermore, we assume that $u_i(t) \in \mathcal{H}_m$ satisfies sufficient regularity conditions such that (4.6) has a unique solution forward in time.

Specifically, we assume that the control process $u_i(\cdot)$ in (4.6) is restricted to the class of *admissible* controls consisting of measurable functions $u_i(\cdot)$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that $u_i(t) \in \mathcal{H}_m$ and, for all $t \geq s$, $w(t) - w(s)$ is independent of $u_i(\tau)$, $w(\tau)$, $\tau \leq s$, and $x_0(0)$, and hence, $u_i(\cdot)$ is nonanticipative. In addition, we assume that $u_i(\cdot)$ takes values in a compact metrizable set, and hence, it follows from Theorem 2.2.4 of [4] that there exists a unique pathwise solution to (4.7) in $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}^{x_{i0}})$ for every $i \in \{1, \ldots, N\}$.

Furthermore, we assume that the control input of the ith follower agent with $i \in \mathcal{N}_{\mathrm{I}}$ is compromised and is given by

$$\tilde{u}_i(t) = \Delta_i(t)u_i(t) + d_i(t), \quad i \in \mathcal{N}_{\mathrm{I}}, \tag{4.3}$$

where, for $t \geq 0$ and $i \in \mathcal{N}_{\mathrm{I}}$, $\tilde{u}_i(t) \in \mathcal{H}_m$ denotes the compromised control signal, $\Delta_i(t) = \mathrm{diag}[\delta_{i1}(t), \ldots, \delta_{im}(t)] \in \mathbb{R}^{m\times m}$, where $\delta_{ik}(t) \in \mathbb{R}$, $k \in \{1, \ldots, m\}$, represents a multiplicative actuator attack such that $0 < \delta_{ik,\min} \leq \delta_{ik}(t) \leq \delta_{ik,\max}$ with $\delta_{ik,\min}$ and $\delta_{ik,\max}$ denoting upper and lower bounds, respectively, and $d_i(t) \in \mathbb{R}^m$ denotes an additive actuator attack. Moreover, the control input of the ith follower agent with $i \in \mathcal{N}_{\mathrm{II}}$ is compromised and is given by

$$\tilde{u}_i(t) = u_i(t) + d_i(t), \quad i \in \mathcal{N}_{\mathrm{II}}, \tag{4.4}$$

where, for $t \geq 0$ and $i \in \mathcal{N}_{\mathrm{II}}$, $d_i(t) \in \mathbb{R}^m$ represents an additive actuator attack.

Note that (4.3) and (4.4) can be combined as

$$\tilde{u}_i(t) = \Delta_i(t)u_i(t) + d_i(t), \quad i = 1, \ldots, N, \tag{4.5}$$

where, for $i \in \mathcal{N}_{\mathrm{II}}$, we take $\Delta_i(t) \equiv I_m$. Now, the compromised controlled system is given by

$$\mathrm{d}x_i(t) = [Ax_i(t) + B\tilde{u}_i(t)]\mathrm{d}t + x_i(t)g^{\mathrm{T}}\mathrm{d}w(t), \quad x_i(0) \overset{\text{a.s.}}{=} x_{i0}, \quad i = 1, \ldots, N, \quad t \geq 0. \tag{4.6}$$

Next, the leader dynamics are given by

$$\mathrm{d}x_0(t) = [Ax_0(t) + Br_0(t)]\mathrm{d}t + x_0(t)g^{\mathrm{T}}\mathrm{d}w(t), \quad x_0(0) \overset{\text{a.s.}}{=} x_{00}, \quad t \geq 0, \tag{4.7}$$

where, for $t \geq 0$, $x_0(t) \in \mathcal{H}_n$ is the leader state and $r_0(t) \in \mathbb{R}^m$ is a bounded continuous reference input. Here, we assume that $r_0(\cdot)$ satisfies sufficient regularity conditions such that (4.7) has a unique solution forward in time.

In the literature, the leader-follower consensus problem formulation typically assumes a relative state information between neighbouring agents in order to derive the ith agent controller. Specifically, for $i \in \{1, \ldots, N\}$, the neighbourhood synchronization error [29, 41, 108, 131, 132, 145, 154, 158, 159] is given by

$$\bar{e}_i(t) = \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} \mathcal{A}_{(i,j)}[x_i(t) - x_j(t)] + \mathcal{A}_{(i,0)}[x_i(t) - x_0(t)]. \tag{4.8}$$

Note that the structure of the neighbourhood synchronization error given by (4.8) assumes exact measurement of the leader information $x_0(t)$, $t \geq 0$, by the ith follower agent for every $i \in \mathcal{N}_{\mathrm{II}}$. However, this may not always be the case in practice. Specifically, in the case where we have communication channel attacks or when the sensors measuring the leader state are under attack, the leader state information $x_0(t)$, $t \geq 0$, may not be accurately available to the agents. A more realistic scenario is thus the case where $x_{0i,\mathrm{m}}(t) \overset{\text{a.s.}}{\neq} x_0(t)$, where $x_{0i,\mathrm{m}}(t)$, $t \geq 0$, is the leader state information measured or received by the ith follower agent for $i \in \mathcal{N}_{\mathrm{II}}$.

In this case, for $i \in \mathcal{N}_{\mathrm{II}}$, the compromised leader measurement by the ith agent is given by

$$x_{0i,\mathrm{m}}(t) = \Theta_i(t)x_0(t), \tag{4.9}$$

where $\Theta_i(t) = \mathrm{diag}[\theta_{i1}(t), \ldots, \theta_{in}(t)] \in \mathbb{R}^{n \times n}$ with $\theta_{ik}(t) \neq 0$, $k \in \{1, \ldots, n\}$, and all $t \geq 0$. Note that for generality we assume $x_{0i,\mathrm{m}}(t) \overset{\text{a.s.}}{\neq} x_{0j,\mathrm{m}}(t)$, $i, j \in \mathcal{N}_{\mathrm{II}}$, $i \neq j$. For agent $i \in \mathcal{N}_{\mathrm{I}}$, $\mathcal{A}_{(i,0)} = 0$, which implies that agent i does not have access to the leader information, and hence, $\Theta_i(t)$, $t \geq 0$, is set to $\Theta_i(t) \equiv I_n$.

The following assumptions are necessary for the main results of this section.

Assumption 4.1. The undirected communication graph topology $\mathfrak{G}$ is connected and at least one follower agent is connected to the leader.

Assumption 4.1 implies that $\mathfrak{L}_1$ is symmetric and positive definite [77, 145, 154].

Assumption 4.2. For $t \geq 0$ and $i \in \{1, \ldots, N\}$, there exist *unknown* scalars $\bar{r}_0$, $\bar{x}_0$, $\bar{d}_i$, $\bar{\Theta}_{j,-1}$, $j \in \mathcal{N}_{\mathrm{II}}$, $\bar{\dot{\Theta}}_{j,-1}$, $j \in \mathcal{N}_{\mathrm{II}}$, $\bar{\Delta}_j$, $j \in \mathcal{N}_{\mathrm{I}}$, and $\bar{\dot{\Delta}}_j$, $j \in \mathcal{N}_{\mathrm{I}}$, such that $\|r_0(t)\| \leq \bar{r}_0$, $\|x_0(t)\| \leq \bar{x}_0$, $\|d_i(t)\| \leq \bar{d}_i$, $\|\Theta_j^{-1}(t)\|_{\mathrm{F}} \leq \bar{\Theta}_{j,-1}$, $j \in \mathcal{N}_{\mathrm{II}}$, $\|\dot{\Theta}_j^{-1}(t)\|_{\mathrm{F}} \leq \bar{\dot{\Theta}}_{j,-1}$, $j \in \mathcal{N}_{\mathrm{II}}$, $\|\Delta_j(t)\|_{\mathrm{F}} \leq \bar{\Delta}_j$, $j \in \mathcal{N}_{\mathrm{I}}$, and $\|\dot{\Delta}_j(t)\|_{\mathrm{F}} \leq \bar{\dot{\Delta}}_j$, $j \in \mathcal{N}_{\mathrm{I}}$.

4.4. Distributed Adaptive Control Design

In this section, we develop a distributed adaptive control architecture for the stochastic multiagent system given by (4.6) and (4.7). The control action for the ith follower agent is given by

$$u_i(t) = \hat{\Delta}_i^{-1}(t) u_{i0}(t), \tag{4.10}$$

$$u_{i0}(t) = -cKe_i(t), \tag{4.11}$$

where $c > 0$ is a design constant, $K \in \mathbb{R}^{m \times n}$ is a control gain to be determined, $\hat{\Delta}_i(t) \equiv I_m$, $i \in \mathcal{N}_{\mathrm{II}}$, and $\hat{\Delta}_i(t) \in \mathcal{H}_{m \times m}$, $i \in \mathcal{N}_{\mathrm{I}}$, $t \geq 0$, are the estimates of $\Delta_i(t)$, $i \in \mathcal{N}_{\mathrm{I}}$, $t \geq 0$. In light of the fact that we do not assume an exact measurement for the leader information by the follower agents that are in direct communication with the leader, we formulate a new neighbourhood synchronization error $e_i(t)$, $i \in \{1, \ldots, N\}$, $t \geq 0$, given by

$$e_i(t) = \sum_{j \in \mathcal{N}_{\mathrm{in}}(i)} \mathcal{A}_{(i,j)}[x_i(t) - x_j(t)] + \mathcal{A}_{(i,0)}[x_i(t) - \hat{\Upsilon}_i(t) x_{0i,\mathrm{m}}(t)], \tag{4.12}$$

where $\hat{\Upsilon}_i(t) \equiv I_n$, $i \in \mathcal{N}_{\mathrm{I}}$, and $\hat{\Upsilon}_i(t) \in \mathcal{H}_{n \times n}$, $i \in \mathcal{N}_{\mathrm{II}}$, $t \geq 0$, is the estimate of $\Theta_i^{-1}(t)$, $i \in \mathcal{N}_{\mathrm{II}}$, $t \geq 0$.

The update laws $\hat{\Upsilon}_i(t) \in \mathcal{H}_{n\times n}$, $i \in \mathcal{N}_{\rm II}$, $t \geq 0$, and $\hat{\Delta}_i(t) \in \mathcal{H}_{m\times m}$, $i \in \mathcal{N}_{\rm I}$, $t \geq 0$, are given by

$$\mathrm{d}\hat{\Upsilon}_i(t) = -[n_{\Upsilon_i} K^{\rm T} K e_i(t) x_{0i,\rm m}^{\rm T}(t) + n_{\Upsilon_i} \mathcal{A}_{(i,0)} K^{\rm T} K \hat{\Upsilon}_i(t) x_{0i,\rm m}(t) x_{0i,\rm m}^{\rm T}(t) + \sigma_{\Upsilon_i} \hat{\Upsilon}_i(t)]\mathrm{d}t,$$

$$\hat{\Upsilon}_i(0) \overset{\rm a.s.}{=} \hat{\Upsilon}_{i0}, \quad i \in \mathcal{N}_{\rm II}, \quad t \geq 0, \tag{4.13}$$

$$\mathrm{d}\hat{\delta}_{ik}(t) = \begin{cases} 0, & \hat{\delta}_{ik} = \delta_{ik,\min} \text{ and } \phi_{ik}(t) < 0; \\ \phi_{ik}(t)\mathrm{d}t, & \text{otherwise}, \end{cases}$$

$$\hat{\delta}_{ik}(0) \overset{\rm a.s.}{=} \hat{\delta}_{ik0} > \delta_{ik,\min}, \quad i \in \mathcal{N}_{\rm I}, \quad k = 1, \ldots, m, \quad t \geq 0, \tag{4.14}$$

where $\phi_{ik}(t) \triangleq n_{\Delta_i} [e_i^{\rm T}(t) K^{\rm T}]_k [u_i(t)]_k - \sigma_{\Delta_i} \hat{\delta}_{ik}(t)$, $[\,\cdot\,]_k$ denotes the kth component of a vector $[\,\cdot\,]$, $n_{\Upsilon_i} > 0$, $i \in \mathcal{N}_{\rm II}$, $n_{\Delta_i} > 0$, $i \in \mathcal{N}_{\rm I}$, $\sigma_{\Upsilon_i} > 2$, $i \in \mathcal{N}_{\rm II}$, and $\sigma_{\Delta_i} > 1$, $i \in \mathcal{N}_{\rm I}$, are design gains. Note that

$$\begin{aligned} e_i(t) &= \sum_{j\in\mathcal{N}_{\rm in}(i)} \mathcal{A}_{(i,j)}[x_i(t) - x_j(t)] + \mathcal{A}_{(i,0)}[x_i(t) - \hat{\Upsilon}_i(t) x_{0i,\rm m}(t)] \\ &= \sum_{j\in\mathcal{N}_{\rm in}(i)} \mathcal{A}_{(i,j)}[x_i(t) - x_j(t)] + \mathcal{A}_{(i,0)}[x_i(t) - x_0(t)] + \mathcal{A}_{(i,0)}[\Theta_i^{-1}(t) - \hat{\Upsilon}_i(t)] x_{0i,\rm m}(t) \\ &= \sum_{j\in\mathcal{N}_{\rm in}(i)} \mathcal{A}_{(i,j)}[x_i(t) - x_j(t)] + \mathcal{A}_{(i,0)}[x_i(t) - x_0(t)] - \mathcal{A}_{(i,0)} \tilde{\Upsilon}_i(t) x_{0i,\rm m}(t) \\ &= \bar{e}_i(t) - \mathcal{A}_{(i,0)} \tilde{\Upsilon}_i(t) x_{0i,\rm m}(t), \end{aligned} \tag{4.15}$$

where $\tilde{\Upsilon}_i(t) \triangleq \hat{\Upsilon}_i(t) - \Theta_i^{-1}(t)$, $i \in \mathcal{N}_{\rm II}$. By definition $\tilde{\Upsilon}_i(t) \equiv 0$ when $\mathcal{A}_{(i,0)} = 0$, $i \in \mathcal{N}_{\rm I}$, and hence, in this case $e_i(t) = \bar{e}_i(t) = \sum_{j\in\mathcal{N}_{\rm in}(i)} \mathcal{A}_{(i,j)}[x_i(t) - x_j(t)]$.

Next, it follows from (4.10) that

$$\begin{aligned} \Delta_i(t) u_i(t) &= \hat{\Delta}_i(t) \hat{\Delta}_i^{-1}(t) u_{i0}(t) - \hat{\Delta}_i(t) \hat{\Delta}_i^{-1}(t) u_{i0}(t) + \Delta_i(t) u_i(t) \\ &= u_{i0}(t) - \tilde{\Delta}_i(t) u_i(t), \quad i = 1, \ldots, N, \end{aligned} \tag{4.16}$$

where $\tilde{\Delta}_i(t) \triangleq \hat{\Delta}_i(t) - \Delta_i(t)$. Furthermore, by definition $\tilde{\Delta}_i(t) \equiv 0$, $i \in \mathcal{N}_{\rm II}$. Now, defining the tracking error $\varepsilon_i(t) \triangleq x_i(t) - x_0(t)$ and using (4.10) and (4.11), the dynamics for the tracking error of the ith agent is given by

$$\mathrm{d}\varepsilon_i(t) = [A\varepsilon_i(t) - cBKe_i(t) - B\tilde{\Delta}_i(t) u_i(t) + B(d_i(t) - r_0(t))]\mathrm{d}t + \varepsilon_i(t) g^{\rm T} \mathrm{d}w(t),$$

$$\varepsilon_i(0) \overset{\text{a.s.}}{=} \varepsilon_{i0}, \quad t \geq 0. \tag{4.17}$$

For the statement of the next result, sgn denotes the sign operator, that is, $\mathrm{sgn}(\alpha) \triangleq \frac{\alpha}{|\alpha|}$, $\alpha \neq 0$, and $\mathrm{sgn}(0) \triangleq 0$. Furthermore, by Assumption 4.2, there exist constants d_{i1}, $i \in \{1, \ldots, N\}$, such that $\|d_i(t) - r_0(t)\| \leq d_{i1}$, $i \in \{1, \ldots, N\}$, $t \geq 0$, and, for every finite $K \in \mathbb{R}^{m \times n}$, there exist constants $d_{i2} > 0$, $i \in \mathcal{N}_{\mathrm{II}}$, such that $|\mathrm{tr}[K^{\mathrm{T}} K \Theta_i^{-1}(t) x_{0i,\mathrm{m}}(t) x_{0i,\mathrm{m}}^{\mathrm{T}}(t)]| \leq d_{i2}$, $i \in \mathcal{N}_{\mathrm{II}}$, $t \geq 0$. Finally, by definition, $\bar{\Theta}_{i,-1} = 1$, $i \in \mathcal{N}_{\mathrm{I}}$, $\dot{\bar{\Theta}}_{i,-1} = 0$, $i \in \mathcal{N}_{\mathrm{I}}$, $\bar{\Delta}_i = 1$, $i \in \mathcal{N}_{\mathrm{II}}$, and $\dot{\bar{\Delta}}_i = 0$, $i \in \mathcal{N}_{\mathrm{II}}$.

Theorem 4.1. Consider the stochastic multiagent system given by (4.6) and (4.7) with actuator and sensor attacks given by (4.5) and (4.9), respectively. Assume Assumptions 4.1 and 4.2 hold, and, for a given positive-definite matrix $R \in \mathbb{R}^{n \times n}$, assume there exists a positive-definite matrix $P \in \mathbb{R}^{n \times n}$ such that

$$\tilde{A}^{\mathrm{T}} P + P \tilde{A} - 2(c - \gamma_1)\lambda_{\min}(\mathfrak{L}_1) P B B^{\mathrm{T}} P + R = 0, \tag{4.18}$$

where $\tilde{A} \triangleq A + \frac{1}{2}\|g\|^2 I_n$. Then, with the controller given by (4.10) and (4.11), adaptive laws given by (4.13) and (4.14), and control gain $K = B^{\mathrm{T}} P$, the closed-loop system given by (4.13), (4.14), and (4.17) satisfies

$$\limsup_{t \to \infty} \mathbb{E}^{\varepsilon_{i0}}[\|x_i(t) - x_0(t)\|^2] \leq \frac{c_0}{c_1 \lambda_{\min}(\mathfrak{L}_1 \otimes P)}, \quad i = 1, \ldots, N, \tag{4.19}$$

where

$$\begin{aligned} c_0 \triangleq{}& \sum_{i=1}^{N} \frac{(1 - \mathrm{sgn}(\mathcal{A}_{(i,0)}))\sigma_{\Delta_i}}{n_{\Delta_i}} \bar{\Delta}_i^2 + \sum_{i=1}^{N} \frac{1 - \mathrm{sgn}(\mathcal{A}_{(i,0)})}{n_{\Delta_i}} \dot{\bar{\Delta}}_i^2 + \sum_{i=1}^{N} \frac{\mathcal{A}_{(i,0)} c \sigma_{\Upsilon_i}}{n_{\Upsilon_i}} \bar{\Theta}_{i,-1}^2 \\ &+ \sum_{i=1}^{N} \frac{\mathcal{A}_{(i,0)} c}{n_{\Upsilon_i}} \dot{\bar{\Theta}}_{i,-1}^2 + \sum_{i=1}^{N} \frac{1}{\gamma_1} d_{i1}^2 + \sum_{i=1}^{N} \mathcal{A}_{(i,0)}^3 c n_{\Upsilon_i} d_{i2}^2 \end{aligned}$$

and

$$c_1 \triangleq \min\left\{\sigma_{\Upsilon_1} - 2, \ldots, \sigma_{\Upsilon_N} - 2, \sigma_{\Delta_1} - 1, \ldots, \sigma_{\Delta_N} - 1, \frac{\lambda_{\min}(\mathfrak{L}_1)\lambda_{\min}(R)}{\lambda_{\max}(\mathfrak{L}_1 \otimes P)}\right\}.$$

Furthermore, the adaptive estimates $\hat{\Upsilon}_i(t)$, $i \in \mathcal{N}_{\mathrm{II}}$, $t \geq 0$, and $\hat{\Delta}_i(t)$, $i \in \mathcal{N}_{\mathrm{I}}$, $t \geq 0$, are ultimately uniformly bounded in a mean-square sense.

Proof: To show ultimate boundedness of the closed-loop system, consider the Lyapunov-like function given by

$$V\big(\varepsilon, \tilde{\Delta}, \tilde{\Upsilon}\big) = \varepsilon^{\mathrm{T}}(\mathfrak{L}_1 \otimes P)\varepsilon + \sum_{i=1}^{N} \frac{1-\mathrm{sgn}(\mathcal{A}_{(i,0)})}{n_{\Delta_i}}\mathrm{tr}(\tilde{\Delta}_i^2) + \sum_{i=1}^{N} \frac{\mathcal{A}_{(i,0)}c}{n_{\Upsilon_i}}\mathrm{tr}(\tilde{\Upsilon}_i^{\mathrm{T}}\tilde{\Upsilon}_i), \quad (4.20)$$

where $\varepsilon = [\varepsilon_1^{\mathrm{T}}, \ldots, \varepsilon_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{nN}$, $\tilde{\Delta} = \text{block-diag}[\tilde{\Delta}_1, \ldots, \tilde{\Delta}_N] \in \mathbb{R}^{nN\times nN}$, $\tilde{\Upsilon} =$ block-diag$[\tilde{\Upsilon}_1, \ldots,$

$\tilde{\Upsilon}_N] \in \mathbb{R}^{nN\times nN}$, and P satisfies (4.18). Note that if $i \in \mathcal{N}_{\mathrm{I}}$, then $\mathcal{A}_{(i,0)} = 0$, and hence, $\frac{1-\mathrm{sgn}(\mathcal{A}_{(i,0)})}{n_{\Delta_i}}\mathrm{tr}(\tilde{\Delta}_i^2) = \frac{1}{n_{\Delta_i}}\mathrm{tr}(\tilde{\Delta}_i^2)$. In addition, $\frac{\mathcal{A}_{(i,0)}c}{n_{\Upsilon_i}}\mathrm{tr}(\tilde{\Upsilon}_i^{\mathrm{T}}\tilde{\Upsilon}_i) = 0$. Alternatively, if $i \in \mathcal{N}_{\mathrm{II}}$, then $\mathcal{A}_{(i,0)} > 0$, and hence, $\frac{1-\mathrm{sgn}(\mathcal{A}_{(i,0)})}{n_{\Delta_i}}\mathrm{tr}(\tilde{\Delta}_i^2) = 0$. In this case, $\frac{\mathcal{A}_{(i,0)}c}{n_{\Upsilon_i}}\mathrm{tr}(\tilde{\Upsilon}_i^{\mathrm{T}}\tilde{\Upsilon}_i) \neq 0$.

Now, the infinitesimal generator $\mathcal{L}V\big(\varepsilon, \tilde{\Delta}, \tilde{\Upsilon}\big)$ of the closed-loop system (4.13), (4.14), and (4.17) is given by

$$\begin{aligned}\mathcal{L}V\big(\varepsilon, \tilde{\Delta}, \tilde{\Upsilon}\big) &= 2\varepsilon^{\mathrm{T}}(\mathfrak{L}_1 \otimes PA - c\mathfrak{L}_1^2 \otimes PBB^{\mathrm{T}}P)\varepsilon + 2c\sum_{i=1}^{N} \bar{e}_i^{\mathrm{T}} PBB^{\mathrm{T}}P\mathcal{A}_{(i,0)}\tilde{\Upsilon}_i x_{0i,\mathrm{m}} \\ &\quad + 2\sum_{i=1}^{N} \bar{e}_i^{\mathrm{T}} PB(d_i - r_0) - 2\sum_{i=1}^{N} \bar{e}_i^{\mathrm{T}} PB\tilde{\Delta}_i u_i + \mathrm{tr}[g\varepsilon^{\mathrm{T}}\left(\mathfrak{L}_1 \otimes P\right)\varepsilon g^{\mathrm{T}}] \\ &\quad + \sum_{i=1}^{N} \frac{2(1-\mathrm{sgn}(\mathcal{A}_{(i,0)}))}{n_{\Delta_i}}\mathrm{tr}[\tilde{\Delta}_i(n_{\Delta_i}u_i e_i^{\mathrm{T}} PB - \sigma_{\Delta_i}\hat{\Delta}_i - \dot{\Delta}_i)] \\ &\quad + \sum_{i=1}^{N} \frac{2\mathcal{A}_{(i,0)}c}{n_{\Upsilon_i}}\mathrm{tr}[\tilde{\Upsilon}_i^{\mathrm{T}}(-n_{\Upsilon_i}PBB^{\mathrm{T}}Pe_i x_{0i,\mathrm{m}}^{\mathrm{T}} - n_{\Upsilon_i}\mathcal{A}_{(i,0)}PBB^{\mathrm{T}}P\hat{\Upsilon}_i x_{0i,\mathrm{m}}x_{0i,\mathrm{m}}^{\mathrm{T}} \\ &\quad - \sigma_{\Upsilon_i}\hat{\Upsilon}_{ik} - \dot{\Theta}_i^{-1})], \quad (\varepsilon, \tilde{\Delta}, \tilde{\Upsilon}) \in \mathbb{R}^{Nn} \times \mathbb{R}^{Nm\times Nm} \times \mathbb{R}^{Nn\times Nn}. \quad (4.21)\end{aligned}$$

For $\gamma_1 > 0$, note that

$$\begin{aligned} 2\sum_{i=1}^{N} \bar{e}_i^{\mathrm{T}} PB(d_i - r_0) &\leq \sum_{i=1}^{N} \gamma_1 \bar{e}_i^{\mathrm{T}} PBB^{\mathrm{T}}P\bar{e}_i + \sum_{i=1}^{N} \frac{1}{\gamma_1}(d_i - r_0)^{\mathrm{T}}(d_i - r_0) \\ &\leq \sum_{i=1}^{N} \gamma_1 \bar{e}_i^{\mathrm{T}} PBB^{\mathrm{T}}P\bar{e}_i + \sum_{i=1}^{N} \frac{1}{\gamma_1} d_{i1}^2 \\ &= \gamma_1\varepsilon^{\mathrm{T}}(\mathfrak{L}_1 \otimes I_n)\left(I_N \otimes PBB^{\mathrm{T}}P\right)(\mathfrak{L}_1 \otimes I_n)\varepsilon + \sum_{i=1}^{N} \frac{1}{\gamma_1} d_{i1}^2 \\ &= \gamma_1\varepsilon^{\mathrm{T}}(\mathfrak{L}_1^2 \otimes PBB^{\mathrm{T}}P)\varepsilon + \sum_{i=1}^{N} \frac{1}{\gamma_1} d_{i1}^2. \quad (4.22)\end{aligned}$$

Next, using the first part of (4.14) we have $\tilde{\delta}_{ik} \leq 0$ and $n_{\Delta_i}[e_i^{\mathrm{T}} PB]_k [u_i]_k < \sigma_{\Delta_i} \hat{\delta}_{ik}$, $i \in \mathcal{N}_{\mathrm{I}},\ k = 1, \ldots, m$, and hence,

$$-2[e_i^{\mathrm{T}} PB]_k [u_i]_k \tilde{\delta}_{ik} < -\frac{2\sigma_{\Delta_i}(1 - \operatorname{sgn}(\mathcal{A}_{(i,0)}))}{n_{\Delta_i}} \tilde{\delta}_{ik} \hat{\delta}_{ik}. \tag{4.23}$$

Alternatively, using the second part of (4.14) we have

$$-2 \sum_{i=1}^{N} \bar{e}_i^{\mathrm{T}} PB \tilde{\Delta}_i u_i = -\sum_{i=1}^{N} \frac{2(1 - \operatorname{sgn}(\mathcal{A}_{(i,0)}))}{n_{\Delta_i}} \operatorname{tr}\left(\tilde{\Delta}_i n_{\Delta_i} u_i e_i^{\mathrm{T}} PB\right). \tag{4.24}$$

Furthermore, note that

$$\begin{aligned} 2c \sum_{i=1}^{N} \bar{e}_i^{\mathrm{T}} PBB^{\mathrm{T}} P \mathcal{A}_{(i,0)} \tilde{\Upsilon}_i x_{0i,\mathrm{m}} &= 2c \sum_{i=1}^{N} (e_i + \mathcal{A}_{(i,0)} \tilde{\Upsilon}_i x_{0i,\mathrm{m}})^{\mathrm{T}} PBB^{\mathrm{T}} P \mathcal{A}_{(i,0)} \tilde{\Upsilon}_i x_{0i,\mathrm{m}} \\ &= 2c \sum_{i=1}^{N} e_i^{\mathrm{T}} PBB^{\mathrm{T}} P \mathcal{A}_{(i,0)} \tilde{\Upsilon}_i x_{0i,\mathrm{m}} \\ &\quad + 2c \sum_{i=1}^{N} (\mathcal{A}_{(i,0)} \tilde{\Upsilon}_i x_{0i,\mathrm{m}})^{\mathrm{T}} PBB^{\mathrm{T}} P \mathcal{A}_{(i,0)} \tilde{\Upsilon}_i x_{0i,\mathrm{m}}. \end{aligned} \tag{4.25}$$

Now, using the fact that $\operatorname{tr}(Q^{\mathrm{T}} v y^{\mathrm{T}}) = \operatorname{tr}(Q^{\mathrm{T}} v y^{\mathrm{T}})^{\mathrm{T}} = v^{\mathrm{T}} Q y$ for every $Q \in \mathbb{R}^{m \times n}$, $v \in \mathbb{R}^m$, and $y \in \mathbb{R}^N$, it can be shown that

$$2c \sum_{i=1}^{N} e_i^{\mathrm{T}} PBB^{\mathrm{T}} P \mathcal{A}_{(i,0)} \tilde{\Upsilon}_i x_{0i,\mathrm{m}} = \sum_{i=1}^{N} \frac{2\mathcal{A}_{(i,0)} c}{n_{\Upsilon_i}} \operatorname{tr}\left(\tilde{\Upsilon}_i^{\mathrm{T}} n_{\Upsilon_i} PBB^{\mathrm{T}} P e_i x_{0i,\mathrm{m}}^{\mathrm{T}}\right) \tag{4.26}$$

and

$$\begin{aligned} &2c \sum_{i=1}^{N} (\mathcal{A}_{(i,0)} \tilde{\Upsilon}_i x_{0i,\mathrm{m}})^{\mathrm{T}} PBB^{\mathrm{T}} P \mathcal{A}_{(i,0)} \tilde{\Upsilon}_i x_{0i,\mathrm{m}} \\ &\qquad = \sum_{i=1}^{N} \frac{2\mathcal{A}_{(i,0)} c}{n_{\Upsilon_i}} \operatorname{tr}\left(\tilde{\Upsilon}_i^{\mathrm{T}} n_{\Upsilon_i} \mathcal{A}_{(i,0)} PBB^{\mathrm{T}} P \hat{\Upsilon}_i x_{0i,\mathrm{m}} x_{0i,\mathrm{m}}^{\mathrm{T}}\right) \\ &\qquad\quad - \sum_{i=1}^{N} \frac{2\mathcal{A}_{(i,0)} c}{n_{\Upsilon_i}} \operatorname{tr}\left(\tilde{\Upsilon}_i^{\mathrm{T}} n_{\Upsilon_i} \mathcal{A}_{(i,0)} PBB^{\mathrm{T}} P \Theta_i^{-1} x_{0i,\mathrm{m}} x_{0i,\mathrm{m}}^{\mathrm{T}}\right) \\ &\qquad \leq \sum_{i=1}^{N} \frac{2\mathcal{A}_{(i,0)} c}{n_{\Upsilon_i}} \operatorname{tr}\left(\tilde{\Upsilon}_i^{\mathrm{T}} n_{\Upsilon_i} \mathcal{A}_{(i,0)} PBB^{\mathrm{T}} P \hat{\Upsilon}_i x_{0i,\mathrm{m}} x_{0i,\mathrm{m}}^{\mathrm{T}}\right) \\ &\qquad\quad + \sum_{i=1}^{N} \frac{\mathcal{A}_{(i,0)} c}{n_{\Upsilon_i}} \operatorname{tr}\left(\tilde{\Upsilon}_i^{\mathrm{T}} \tilde{\Upsilon}_i\right) + \sum_{i=1}^{N} \mathcal{A}_{(i,0)}^3 c n_{\Upsilon_i} d_{i2}^2. \end{aligned} \tag{4.27}$$

Next, noting that

$$
\begin{aligned}
-\sum_{i=1}^{N}&\frac{2\mathcal{A}_{(i,0)}\sigma_{\Upsilon_i}c}{n_{\Upsilon_i}}\mathrm{tr}(\tilde{\Upsilon}_i^{\mathrm{T}}\hat{\Upsilon}_i)\\
&=-\sum_{i=1}^{N}\frac{2\mathcal{A}_{(i,0)}\sigma_{\Upsilon_i}c}{n_{\Upsilon_i}}\mathrm{tr}[\tilde{\Upsilon}_i^{\mathrm{T}}(\tilde{\Upsilon}_i+\Theta_i^{-1})]\\
&\leq-\sum_{i=1}^{N}\frac{2\mathcal{A}_{(i,0)}\sigma_{\Upsilon_i}c}{n_{\Upsilon_i}}\mathrm{tr}(\tilde{\Upsilon}_i^{\mathrm{T}}\tilde{\Upsilon}_i)+\sum_{i=1}^{N}\frac{\mathcal{A}_{(i,0)}\sigma_{\Upsilon_i}c}{n_{\Upsilon_i}}\mathrm{tr}(\tilde{\Upsilon}_i^{\mathrm{T}}\tilde{\Upsilon}_i)+\sum_{i=1}^{N}\frac{\mathcal{A}_{(i,0)}\sigma_{\Upsilon_i}c}{n_{\Upsilon_i}}\mathrm{tr}(\Theta_i^{-2})\\
&=-\sum_{i=1}^{N}\frac{\mathcal{A}_{(i,0)}\sigma_{\Upsilon_i}c}{n_{\Upsilon_i}}\mathrm{tr}(\tilde{\Upsilon}_i^{\mathrm{T}}\tilde{\Upsilon}_i)+\sum_{i=1}^{N}\frac{\mathcal{A}_{(i,0)}\sigma_{\Upsilon_i}c}{n_{\Upsilon_i}}\mathrm{tr}(\Theta_i^{-2}) \qquad (4.28)
\end{aligned}
$$

and

$$
-\sum_{i=1}^{N}\frac{2\mathcal{A}_{(i,0)}c}{n_{\Upsilon_i}}\mathrm{tr}(\tilde{\Upsilon}_i^{\mathrm{T}}\dot{\Theta}_i^{-1})\leq\sum_{i=1}^{N}\frac{\mathcal{A}_{(i,0)}c}{n_{\Upsilon_i}}\mathrm{tr}(\tilde{\Upsilon}_i^{\mathrm{T}}\tilde{\Upsilon}_i)+\sum_{i=1}^{N}\frac{\mathcal{A}_{(i,0)}c}{n_{\Upsilon_i}}\mathrm{tr}(\dot{\Theta}_i^{-2}), \qquad (4.29)
$$

it follows that

$$
\begin{aligned}
-\sum_{i=1}^{N}\frac{2\mathcal{A}_{(i,0)}c}{n_{\Upsilon_i}}\mathrm{tr}\left(\tilde{\Upsilon}_i^{\mathrm{T}}(\sigma_{\Upsilon_i}\hat{\Upsilon}_{ik}+\dot{\Theta}_i^{-1})\right)\leq&-\sum_{i=1}^{N}\frac{\mathcal{A}_{(i,0)}c(\sigma_{\Upsilon_i}-1)}{n_{\Upsilon_i}}\mathrm{tr}(\tilde{\Upsilon}_i^{\mathrm{T}}\tilde{\Upsilon}_i)\\
&+\sum_{i=1}^{N}\frac{\mathcal{A}_{(i,0)}c\sigma_{\Upsilon_i}}{n_{\Upsilon_i}}\mathrm{tr}(\Theta_i^{-2})+\sum_{i=1}^{N}\frac{\mathcal{A}_{(i,0)}c}{n_{\Upsilon_i}}\mathrm{tr}(\dot{\Theta}_i^{-2}).
\end{aligned}
\qquad (4.30)
$$

Using (4.30) and a similar construction as above for bounding $-\sum_{i=1}^{N}\frac{2(1-\mathrm{sgn}(\mathcal{A}_{(i,0)}))}{n_{\Delta_i}}\mathrm{tr}[\tilde{\Delta}_i$ $(\sigma_{\Delta_i}\hat{\Delta}_i+\dot{\Delta}_i)]$, as well as using the fact that $\mathrm{tr}[g\varepsilon^{\mathrm{T}}(\mathfrak{L}_1\otimes P)\,\varepsilon g^{\mathrm{T}}]=\|g\|^2\varepsilon^{\mathrm{T}}[\mathfrak{L}_1\otimes P]\varepsilon$, it can be shown, after some algebraic manipulation, that (4.21) yields

$$
\begin{aligned}
\mathcal{L}V\big(\varepsilon,\tilde{\Delta},\tilde{\Upsilon}\big)\leq\,&2\varepsilon^{\mathrm{T}}[\mathfrak{L}_1\otimes PA-(c-\gamma_1)\mathfrak{L}_1^2\otimes PBB^{\mathrm{T}}P+\|g\|^2\mathfrak{L}_1\otimes P]\varepsilon\\
&-\sum_{i=1}^{N}\frac{\mathcal{A}_{(i,0)}c(\sigma_{\Upsilon_i}-2)}{n_{\Upsilon_i}}\mathrm{tr}(\tilde{\Upsilon}_i^{\mathrm{T}}\tilde{\Upsilon}_i)-\sum_{i=1}^{N}\frac{(1-\mathrm{sgn}(\mathcal{A}_{(i,0)}))(\sigma_{\Delta_i}-1)}{n_{\Delta_i}}\mathrm{tr}(\tilde{\Delta}_i^2)+c_0,\\
&(\varepsilon,\tilde{\Delta},\tilde{\Upsilon})\in\mathbb{R}^{Nn}\times\mathbb{R}^{Nm\times Nm}\times\mathbb{R}^{Nn\times Nn}, \qquad (4.31)
\end{aligned}
$$

where

$$
c_0=\sum_{i=1}^{N}\frac{(1-\mathrm{sgn}(\mathcal{A}_{(i,0)}))\sigma_{\Delta_i}}{n_{\Delta_i}}\bar{\Delta}_i^2+\sum_{i=1}^{N}\frac{1-\mathrm{sgn}(\mathcal{A}_{(i,0)})}{n_{\Delta_i}}\bar{\dot{\Delta}}_i^2+\sum_{i=1}^{N}\frac{\mathcal{A}_{(i,0)}\sigma_{\Upsilon_i}c}{n_{\Upsilon_i}}\bar{\Theta}_{i,-1}^2
$$

$$+\sum_{i=1}^{N}\frac{\mathcal{A}_{(i,0)}c}{n_{\Upsilon_i}}\bar{\dot{\Theta}}_{i,-1}^2+\sum_{i=1}^{N}\frac{1}{\gamma_1}d_i^2+\sum_{i=1}^{N}\mathcal{A}_{(i,0)}^3 cn_{\Upsilon_i}d_{i2}^2. \tag{4.32}$$

Now, since $\mathfrak{L}_1$ is positive definite, there exists an orthogonal matrix $T \in \mathbb{R}^{N\times N}$ such that $T^{\mathrm{T}}\mathfrak{L}_1 T = \mathrm{diag}[\lambda_1, \ldots, \lambda_N]$, where λ_i, $i \in \{1, \ldots, N\}$, are the eigenvalues of $\mathfrak{L}_1$. Defining $\xi \triangleq (T^{\mathrm{T}} \otimes I_n)\varepsilon$, it follows from (4.31) that

$$\begin{aligned}\mathcal{L}V\big(\varepsilon,\tilde{\Delta},\tilde{\Upsilon}\big) \leq{}& \sum_{i=1}^{N}\lambda_i\xi_i^{\mathrm{T}}[P\tilde{A}+\tilde{A}^{\mathrm{T}}P-2(c-\gamma_1)\lambda_{\min}(\mathfrak{L}_1)PBB^{\mathrm{T}}P]\xi_i\\ &-\sum_{i=1}^{N}\frac{\mathcal{A}_{(i,0)}c(\sigma_{\Upsilon_i}-2)}{n_{\Upsilon_i}}\mathrm{tr}(\tilde{\Upsilon}_i^{\mathrm{T}}\tilde{\Upsilon}_i)-\sum_{i=1}^{N}\frac{(1-\mathrm{sgn}(\mathcal{A}_{(i,0)}))(\sigma_{\Delta_i}-1)}{n_{\Delta_i}}\mathrm{tr}(\tilde{\Delta}_i^2)+c_0,\\ \leq{}& -\lambda_{\min}(\mathfrak{L}_1)\lambda_{\min}(R)\varepsilon^{\mathrm{T}}\varepsilon-\sum_{i=1}^{N}\frac{(1-\mathrm{sgn}(\mathcal{A}_{(i,0)}))(\sigma_{\Delta_i}-1)}{n_{\Delta_i}}\mathrm{tr}(\tilde{\Delta}_i^2)\\ &-\sum_{i=1}^{N}\frac{\mathcal{A}_{(i,0)}c(\sigma_{\Upsilon_i}-2)}{n_{\Upsilon_i}}\mathrm{tr}(\tilde{\Upsilon}_i^{\mathrm{T}}\tilde{\Upsilon}_i)+c_0,(\varepsilon,\tilde{\Delta},\tilde{\Upsilon})\in\mathbb{R}^{Nn}\times\mathbb{R}^{Nm\times Nm}\times\mathbb{R}^{Nn\times Nn}.\end{aligned} \tag{4.33}$$

Next, defining $c_1 \triangleq \min\left\{\sigma_{\Upsilon_1}-2,\ldots,\sigma_{\Upsilon_N}-2,\sigma_{\Delta_1}-1,\ldots,\sigma_{\Delta_N}-1,\frac{\lambda_{\min}(\mathfrak{L}_1)\lambda_{\min}(R)}{\lambda_{\max}(\mathfrak{L}_1\otimes P)}\right\}$, it follows from (4.33) that

$$\mathcal{L}V\big(\varepsilon,\tilde{\Delta},\tilde{\Upsilon}\big) \leq -c_1V\big(\varepsilon,\tilde{\Delta},\tilde{\Upsilon}\big)+c_0, \quad (\varepsilon,\tilde{\Delta},\tilde{\Upsilon})\in\mathbb{R}^{Nn}\times\mathbb{R}^{Nm\times Nm}\times\mathbb{R}^{Nn\times Nn}. \tag{4.34}$$

Now, using Lemma 3.1, it follows from (4.34) that

$$0 \leq \mathbb{E}^{\varepsilon_0}[V\big(\varepsilon(t),\tilde{\Delta}(t),\tilde{\Upsilon}(t)\big)] \leq V\big(\varepsilon(0),\tilde{\Delta}(0),\tilde{\Upsilon}(0)\big)e^{-c_1t}+\frac{c_0}{c_1}, \quad t\geq 0, \tag{4.35}$$

and hence, all the signals of the closed-loop system are uniformly ultimately bounded in probability in a mean-square sense. Finally, noting that

$$\limsup_{t\to\infty}\mathbb{E}^{\varepsilon_0}[\varepsilon^{\mathrm{T}}(t)\,(\mathfrak{L}_1\otimes P)\,\varepsilon(t)] \leq \limsup_{t\to\infty}\mathbb{E}^{\varepsilon_0}[V\big(\varepsilon(t),\tilde{\Delta}(t),\tilde{\Upsilon}(t)\big)] \leq \frac{c_0}{c_1}, \tag{4.36}$$

it follows that, for every $i \in \{1, \ldots, N\}$,

$$\limsup_{t\to\infty}\mathbb{E}^{\varepsilon_{i0}}[\|x_i(t)-x_0(t)\|^2] \leq \frac{c_0}{c_1\lambda_{\min}(\mathfrak{L}_1\otimes P)}, \tag{4.37}$$

which implies that the pathwise trajectory of the state tracking error for each agent of the closed-loop system associated with the plant dynamics is uniformly ultimately bounded in a mean-square sense. $\square$

In the absence of any sensor attacks on the follower agents that measure or receive leader state information, we have $x_{0i,\mathrm{m}}(t) = \Theta_i(t)x_0(t)$, $i \in \mathcal{N}_{\mathrm{II}}$, with $\Theta_i(t) \equiv I_n$, $t \geq 0$. In this case, the estimation error $\tilde{\Upsilon}_i(t) = \hat{\Upsilon}_i(t) - \Theta_i^{-1}(t)$, $i \in \mathcal{N}_{\mathrm{II}}$, vanishes, and hence, c_0 in the Theorem 4.1 reduces to $c_0 \triangleq \sum_{i=1}^{N} \frac{(1-\mathrm{sgn}(\mathcal{A}_{(i,0)}))\sigma_{\Delta_i}}{n_{\Delta_i}} \bar{\Delta}_i^2 + \sum_{i=1}^{N} \frac{1-\mathrm{sgn}(\mathcal{A}_{(i,0)})}{n_{\Delta_i}} \bar{\dot{\Delta}}_i^2 + \sum_{i=1}^{N} \frac{1}{\gamma_1} d_{i1}^2$.

Note that n_{Υ_i}, $i \in \mathcal{N}_{\mathrm{II}}$, and n_{Δ_i}, $i \in \mathcal{N}_{\mathrm{I}}$, are design gain parameters used in the adaptive laws (4.13) and (4.14), respectively, and thus, selecting large values of these parameters can introduce transient oscillations in the update law estimates of $\hat{\Upsilon}_i(t)$, $t \geq 0$, and $\hat{\Delta}_i(t)$, $t \geq 0$. This can be remedied by adding a modification term in the update laws to filter out the high frequency content in the control signal while preserving uniform ultimate boundedness in a mean-square sense. This architecture is developed in [155].

4.5. Illustrative Numerical Example

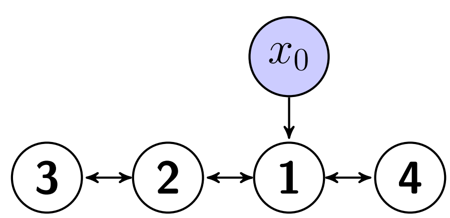


Figure 4.5.1: Leader-follower communication topology of $\mathfrak{G}$.

To illustrate the key ideas presented in this subsection, consider the multiagent system representing the controlled lateral dynamics of four follower aircrafts and one leader aircraft, with a communication topology shown in Figure 4.5.1. Node x_0 represents the leader aircraft and nodes 1 through 4 represent the follower aircrafts. For the leader aircraft, the dynamical system representing the lateral directional dynamics of an aircraft [73] are given by (4.7), where $x_0(t) \triangleq [\beta(t), p(t), r(t)]^{\mathrm{T}}$, $\beta(t)$ is the sideslip angle in deg, $p(t)$ is the roll rate in deg/sec, and $r(t)$ is the yaw rate in deg/sec. Here we take $x_{00} = [1, -2, 1]^{\mathrm{T}}$ and $g = [1,\ 1]^{\mathrm{T}}$. The state-dependent disturbance is used to capture perturbations in atmospheric drag [82]. Furthermore, the system matrices are given by

$$A = \begin{bmatrix} -0.025 & 0.104 & -0.994 \\ 574.7 & 0 & 0 \\ 16.20 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.122 & -0.276 \\ -53.61 & 33.25 \\ 195.5 & -529.4 \end{bmatrix}, \tag{4.38}$$

with reference input

$$r_0(t) = \begin{bmatrix} 2.4056 & 0.0765 & -0.0613 \\ -4.3701 & -0.1086 & 0.1485 \end{bmatrix} x_0(t). \tag{4.39}$$

The follower aircraft dynamics are given by (4.6), where, for $t \geq 0$ and $i \in \{1, 2, 3, 4\}$, $x_1(t) \triangleq [\beta_1(t), p_1(t), r_1(t)]^{\mathrm{T}}$, $x_{10} = [2, -3, -1]^{\mathrm{T}}$, $x_2(t) \triangleq [\beta_2(t), p_2(t), r_2(t)]^{\mathrm{T}}$, $x_{20} = [3, 0, 1]^{\mathrm{T}}$, $x_3(t) \triangleq [\beta_3(t), p_3(t), r_3(t)]^{\mathrm{T}}$, $x_{30} = [2, -1, -1]^{\mathrm{T}}$, and $x_4(t) \triangleq [\beta_4(t), p_4(t), r_4(t)]^{\mathrm{T}}$, $x_{40} = [1, -1, 1.5]^{\mathrm{T}}$. We assume that the leader information received by Agent 1 is given by

$$x_{01,\mathrm{m}}(t) = \begin{bmatrix} 1 + 0.1(1 - e^{-0.5t}) & 0 & 0 \\ 0 & 1 + 0.2(1 - e^{-0.8t}) & 0 \\ 0 & 0 & 1 + 0.4(1 - e^{-0.1t}) \end{bmatrix} x_0(t). \tag{4.40}$$

The uncorrupted control inputs for the follower agents $u_i(t) \triangleq \left[\delta_{\mathrm{ail},i}(t),\ \delta_{\mathrm{rud},i}(t)\right]^{\mathrm{T}}$, $i \in \{1, 2, 3, 4\}$, involve the aileron command in deg and the rudder command in deg, respectively. The actuator attacks are characterized as

$$d_1(t) = \begin{bmatrix} 0.1(1 - e^{-0.1t}) \\ 0.08(1 - e^{-0.15t}) \end{bmatrix}, \quad d_2(t) = \begin{bmatrix} 0.2(1 - e^{-0.1t}) \\ 0.1(1 - e^{-0.15t}) \end{bmatrix}, \tag{4.41}$$

$$d_3(t) = \begin{bmatrix} 0.15(1 - e^{-0.1t}) \\ 0.1(1 - e^{-0.15t}) \end{bmatrix}, \quad d_4(t) = \begin{bmatrix} 0.2(1 - e^{-0.1t}) \\ 0.12(1 - e^{-0.15t}) \end{bmatrix}, \tag{4.42}$$

$$\Delta_2(t) = \begin{bmatrix} 0.8 + 0.2e^{-0.2t} & 0 \\ 0 & 0.85 + 0.15e^{-0.11t} \end{bmatrix}, \tag{4.43}$$

$$\Delta_3(t) = \begin{bmatrix} 0.8 + 0.2e^{-0.15t} & 0 \\ 0 & 0.85 + 0.15e^{-0.15t} \end{bmatrix}, \tag{4.44}$$

$$\Delta_4(t) = \begin{bmatrix} 0.8 + 0.2e^{-0.2t} & 0 \\ 0 & 0.85 + 0.15e^{-0.2t} \end{bmatrix}. \tag{4.45}$$

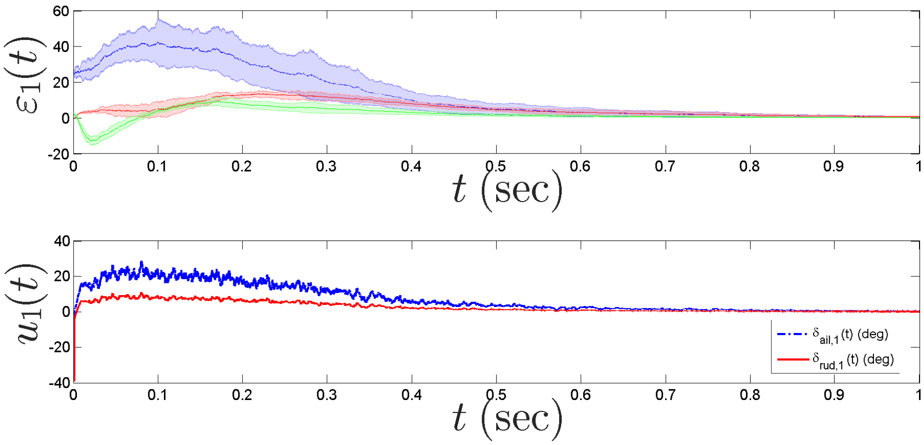


Figure 4.5.2: Agent 1 sample average of the state tracking error profile along with the sample standard deviation of the closed-loop nominal system trajectories versus time; $\beta_1(t) - \beta(t)$ in blue, $p_1(t) - p(t)$ in red, and $r_1(t) - r(t)$ in green. The control profile is plotted as the mean of the 10 sample runs.

Note that at $t = 0$, $\Delta_i(0) = I_2$, $i \in \{2, 3, 4\}$, and $d_i(0) = [0,\ 0]^{\mathrm{T}}$, $i \in \{1, 2, 3, 4\}$, which implies that initially the actuator is uncompromised and is gradually compromised over time.

To design a distributed adaptive controller we use Theorem 4.1 with

$$P = \begin{bmatrix} 56.7970 & 0.1292 & -0.0348 \\ 0.1292 & 0.0411 & 0.0033 \\ -0.0348 & 0.0033 & 0.0031 \end{bmatrix} \tag{4.46}$$

and control design parameters $c = 2$, $\gamma_1 = 0.1$, $n_{\Delta_i} = 1$, $i \in \{2, 3, 4\}$, $\sigma_{\Delta_i} = 2$, $i \in \{2, 3, 4\}$, $n_{\Upsilon_1} = 1$, and $\sigma_{\Upsilon_1} = 3$. The system performance of the controller given by (4.10) and (4.11) with the proposed adaptive scheme is shown in Figures 4.5.2-4.5.5 for the ith follower agent, where $i \in \{1, 2, 3, 4\}$. Specifically, Figures 4.5.2-4.5.5 show a sample trajectory along with the standard deviation of the state tracking error $\varepsilon_i(t) = x_i(t) - x_0(t)$ for agent $i \in \{1, 2, 3, 4\}$ versus time for 10 sample paths. The mean control profile is also plotted in Figures 4.5.2-4.5.5. It follows from Theorem 4.1 that the state tracking error for each agent is guaranteed to be uniformly ultimate bounded in a mean-square sense.

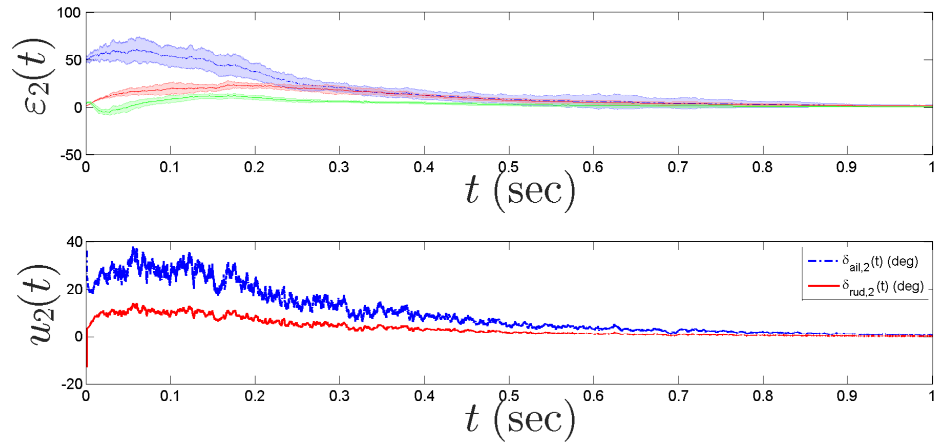


Figure 4.5.3: Agent 2 sample average of the state tracking error profile along with the sample standard deviation of the closed-loop nominal system trajectories versus time; $\beta_2(t) - \beta(t)$ in blue, $p_2(t) - p(t)$ in red, and $r_2(t) - r(t)$ in green. The control profile is plotted as the mean of the 10 sample runs.

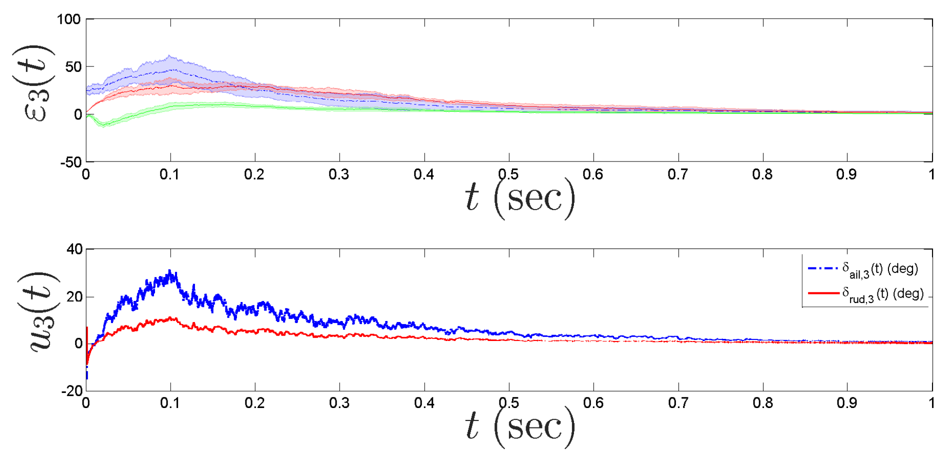


Figure 4.5.4: Agent 3 sample average of the state tracking error profile along with the sample standard deviation of the closed-loop nominal system trajectories versus time; $\beta_3(t) - \beta(t)$ in blue, $p_3(t) - p(t)$ in red, and $r_3(t) - r(t)$ in green. The control profile is plotted as the mean of the 10 sample runs.

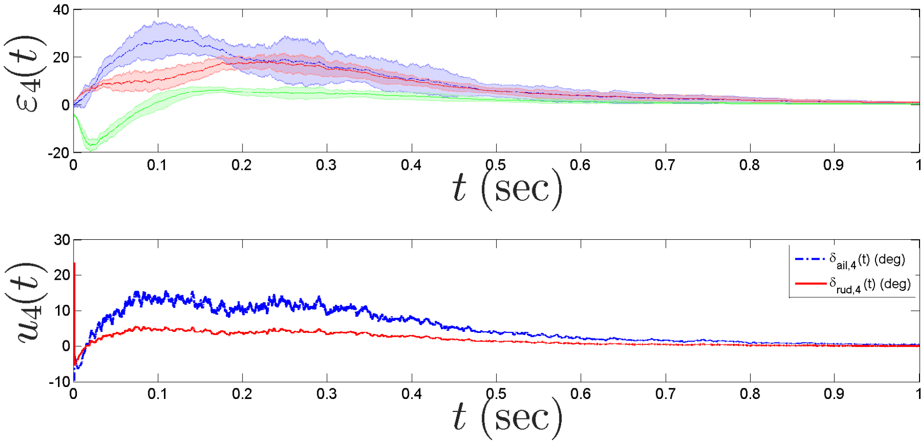


Figure 4.5.5: Agent 4 sample average of the state tracking error profile along with the sample standard deviation of the closed-loop nominal system trajectories versus time; $\beta_4(t) - \beta(t)$ in blue, $p_4(t) - p(t)$ in red, and $r_4(t) - r(t)$ in green. The control profile is plotted as the mean of the 10 sample runs.

4.6. Robust Adaptive Control for Multiagent Systems with Stochastic Disturbances, System Uncertainty, and Sensor-Actuator Attacks

Consider a leader-follower networked multiagent system consisting of N follower agents with the uncertain dynamics of agent $i \in \{1, \ldots, N\}$ given by

$$\mathrm{d}x_i(t) = [(A + \Delta A_i)x_i(t) + Bu_i(t)]\mathrm{d}t + x_i(t)g^{\mathrm{T}}\mathrm{d}w(t), \quad x_i(0) \overset{\text{a.s.}}{=} x_{i0}, \quad t \geq 0, \tag{4.47}$$

where, for $i \in \{1, \ldots, N\}$ and $t \geq 0$, $x_i(t) \in \mathcal{H}_n$ is the state of the ith follower agent, $u_i(t) \in \mathcal{H}_m$ is the uncorrupted control input to the ith follower agent, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are nominal system matrices, $\Delta A_i \in \mathbb{R}^{n\times n}$ characterizes the uncertainty of the ith follower agent dynamics, $w(\cdot)$ is a d-dimensional independent standard Wiener process (i.e., Brownian motion) defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and $g \in \mathbb{R}^d$. Furthermore, we assume that $u_i(t) \in \mathcal{H}_m$ satisfies sufficient regularity conditions such that (4.49) has a unique solution forward in time.

Specifically, we assume that the control process $u_i(\cdot)$ in (4.49) is restricted to the class of *admissible* controls consisting of measurable functions $u_i(\cdot)$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$

such that $u_i(t) \in \mathcal{H}_m$ and, for all $t \geq s$, $w(t) - w(s)$ is independent of $u_i(\tau)$, $w(\tau)$, $\tau \leq s$, and $x_0(0)$, and hence, $u_i(\cdot)$ is nonanticipative. In addition, we assume that $u_i(\cdot)$ takes values in a compact metrizable set, and hence, it follows from Theorem 2.2.4 of [4] that there exists a unique pathwise solution to (4.50) in $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}^{x_{i0}})$ for every $i \in \{1, \ldots, N\}$.

Here, we assume that the control input of the ith follower agent is compromised and is given by

$$\tilde{u}_i(t) = u_i(t) + \phi_i(t), \tag{4.48}$$

where $\tilde{u}_i(t) \in \mathcal{H}_m$, $t \geq 0$, denotes the compromised control signal and $\phi_i(t) \in \mathbb{R}^m$, $t \geq 0$, denotes an additive actuator attack. The compromised controlled uncertain system is given by

$$\mathrm{d}x_i(t) = [(A + \Delta A_i)x_i(t) + B\tilde{u}_i(t)]\mathrm{d}t + x_i(t)g^{\mathrm{T}}\mathrm{d}w(t), \quad x_i(0) \overset{\text{a.s.}}{=} x_{i0}, \quad i = 1, \ldots, N, \quad t \geq 0. \tag{4.49}$$

The leader dynamics are given by

$$\mathrm{d}x_0(t) = [Ax_0(t) + Br_0(t)]\mathrm{d}t + x_0(t)g^{\mathrm{T}}\mathrm{d}w(t), \quad x_0(0) \overset{\text{a.s.}}{=} x_{00}, \quad t \geq 0, \tag{4.50}$$

where $x_0(t) \in \mathcal{H}_n$, $t \geq 0$, is the leader state and $r_0(t) \in \mathbb{R}^m$, $t \geq 0$, is a bounded continuous reference input. Here, we assume that $r_0(\cdot)$ satisfies sufficient regularity conditions such that (4.50) has a unique solution forward in time.

In the literature, the leader-follower consensus problem formulation typically assumes a relative state information between neighbouring agents in order to derive the ith agent controller. Specifically, for $i \in \{1, \ldots, N\}$, the neighbourhood synchronization error [29, 41, 108, 131, 132, 145, 154, 158, 159] is given by

$$\bar{e}_i(t) = \sum_{j\in\mathcal{N}_{\mathrm{in}}(i)} \mathcal{A}_{(i,j)}[x_i(t) - x_j(t)] + \mathcal{A}_{(i,0)}[x_i(t) - x_0(t)]. \tag{4.51}$$

Note that the structure of the neighbourhood synchronization error given by (4.51) assumes exact measurements of the leader information $x_0(t)$, $t \geq 0$, by the ith follower agent, as well as exact measurements of all neighbouring follower agents $x_j(t)$, $t \geq 0$, by the ith follower agent for every $j \in \mathcal{N}_{\text{in}}(i)$. However, this may not always be the case in practice. Specifically, in the case where we have communication channel attacks or when the sensors measuring the leader state as well as the neighbouring follower states are under attack, the neighbourhood synchronization error may not be accurately available to the agents. A more realistic scenario is thus the case where

$$e_i(t) = \bar{e}_i(t) + d_i(t), \tag{4.52}$$

where $d_i(t)$, $i \in \{1, \ldots, N\}$ and $t \geq 0$, captures the uncertainty in the neighbourhood synchronization error quantifying information uncertainty between the follower agents as well as the leader agent.

The attack $d_i(t)$, $i \in \{1, \ldots, N\}$ and $t \geq 0$, can be due to several different sources. For example, if each follower agent state measurement is corrupted, that is,

$$\tilde{x}_i(t) = x_i(t) + \eta_i(t), \tag{4.53}$$

where, for $i \in \{1, \ldots, N\}$ and $t \geq 0$, $\tilde{x}_i(t)$ denotes the corrupted measurement and $\eta_i(t) \in \mathbb{R}^n$ is an additive sensor attack, then

$$d_i(t) = \sum_{j \in \mathcal{N}_{\text{in}}(i)} \mathcal{A}_{(i,j)}[\eta_i(t) - \eta_j(t)] + \mathcal{A}_{(i,0)}\eta_i(t). \tag{4.54}$$

This is a more general sensor attack model than the model considered in [3]. Alternatively, if the leader state measurement is corrupted, that is,

$$\tilde{x}_0(t) = (I_n + \Delta_0(t))x_0(t), \tag{4.55}$$

where $\tilde{x}_0(t)$, $t \geq 0$, is the corrupted measurement of the leader state $x_0(t)$, $t \geq 0$, and $\Delta_0(t) \in \mathbb{R}^{n \times n}$, $t \geq 0$, is a multiplicative sensor attack, then

$$d_i(t) = -\mathcal{A}_{(i,0)}\Delta_0(t)x_0(t). \tag{4.56}$$

The following assumption is necessary for the main results of this subsection.

Assumption 4.3. For $i \in \{1, \dots, N\}$ and $t \geq 0$, there exist a *known* scalar $\bar{\delta}$ and *unknown* scalars $\bar{d}_i$, $\bar{\dot{d}}_i$, $\bar{\theta}_i$, $\bar{\dot{\theta}}_i$, and $\bar{x}_0$, such that $\|\Delta A_i\|_{\mathrm{F}} \leq \bar{\delta}$, $\|\dot{d}_i(t)\| \leq \bar{\dot{d}}_i$, $\|d_i(t)\| \leq \bar{d}_i$, $\|\dot{\theta}_i(t)\| \leq \bar{\dot{\theta}}_i$, $\|\theta_i(t)\| \leq \bar{\theta}_i$, and $\|x_0(t)\| \leq \bar{x}_0$, where $\theta_i(t) \triangleq \phi_i(t) - r_0(t)$.

4.7. Distributed Robust Adaptive Control Design

In this subsection, we develop a distributed adaptive control architecture for the stochastic multiagent system given by (4.49) and (4.50). The control action for the ith follower agent is given by

$$u_i(t) = -cK(e_i(t) - \hat{d}_i(t)) - \hat{\theta}_i(t), \tag{4.57}$$

where $c > 0$ is a design constant, $K \in \mathbb{R}^{m\times n}$ is a control gain to be determined, and, for $i \in \{1, \dots, N\}$ and $t \geq 0$, $\hat{d}_i(t)$ and $\hat{\theta}_i(t)$ are the estimates of $d_i(t)$ and $\theta_i(t)$, respectively.

The update laws $\hat{d}_i(t) \in \mathcal{H}_n$ and $\hat{\theta}_i(t) \in \mathcal{H}_m$ for $t \geq 0$ and $i \in \{1, \dots, N\}$ are given by

$$\mathrm{d}\hat{d}_i(t) = -[2n_{d_i}K^{\mathrm{T}}Ke_i(t) + \sigma_{d_i}\hat{d}(t)]\mathrm{d}t, \quad \hat{d}_i(0) \overset{\text{a.s.}}{=} \hat{d}_{i0}, \quad t \geq 0, \tag{4.58}$$

$$\mathrm{d}\hat{\theta}_i(t) = [2n_{\theta_i}Ke_i(t) - \sigma_{\theta_i}\hat{\theta}_i(t)]\mathrm{d}t, \quad \hat{\theta}_i(0) \overset{\text{a.s.}}{=} \hat{\theta}_{i0}, \tag{4.59}$$

where $n_{d_i} > 0$, $n_{\theta_i} > 0$, $\sigma_{d_i} > 1 + 2n_{d_i}c\gamma_1$, and $\sigma_{\theta_i} > 1 + 2n_{\theta_i}\gamma_4$ are design gains and $\gamma_1 > 0$ and $\gamma_4 > 0$ are constants. Now, defining the tracking error $\varepsilon_i(t) \triangleq x_i(t) - x_0(t)$ and using (4.57), the dynamics for the tracking error of the ith agent are given by

$$\begin{aligned}\mathrm{d}\varepsilon_i(t) = {} & [A\varepsilon_i(t) + \Delta A_i\varepsilon_i(t) + \Delta A_i x_0(t) - cBK\bar{e}_i(t) + cBK\tilde{d}_i(t) - B\tilde{\theta}_i(t)]\mathrm{d}t \\ & + \varepsilon_i(t)g^{\mathrm{T}}\mathrm{d}w(t), \qquad \varepsilon_i(0) \overset{\text{a.s.}}{=} \varepsilon_{i0}, \quad t \geq 0,\end{aligned} \tag{4.60}$$

where $i \in \{1, \dots, N\}$, $\tilde{d}_i(t) \triangleq \hat{d}_i(t) - d_i(t)$, and $\tilde{\theta}_i(t) \triangleq \hat{\theta}_i(t) - \theta_i(t)$. Finally, note that it follows from Assumption 4.3 that, for every finite $K \in \mathbb{R}^{m\times n}$ and $i \in \{1, \dots, N\}$, there exist constants $d_{i1} > 0$ and $d_{i2} > 0$ such that $\|d_i^{\mathrm{T}}(t)K^{\mathrm{T}}K\| \leq d_{i1}$, $t \geq 0$, and $\|Kd_i(t)\| \leq d_{i2}$, $t \geq 0$.

Theorem 4.2. Consider the stochastic multiagent system given by (4.49) and (4.50) with actuator attacks given by (4.48) and neighbourhood synchronization error attacks given by (4.52). Assume Assumptions 4.1 and 4.3 hold, and, for a given positive-definite matrix $R \in \mathbb{R}^{n\times n}$, there exist constants $\gamma_2 > 0$ and $\gamma_3 > 0$ and a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$\tilde{A}^{\mathrm{T}}P + P\tilde{A} - 2c\lambda_{\min}(\mathfrak{L}_1)PBB^{\mathrm{T}}P + (\gamma_2+\gamma_3)\lambda_{\max}(\mathfrak{L}_1)P^2 + \frac{1}{\lambda_{\min}(\mathfrak{L}_1)}\frac{\bar{\delta}^2}{\gamma_2}I_n + R = 0, \tag{4.61}$$

where $\tilde{A} \triangleq A + \frac{1}{2}\|g\|^2 I_n$. Then, with the controller given by (4.57), adaptive laws given by (4.58) and (4.59), and control gain $K = B^{\mathrm{T}}P$, the closed-loop system given by (4.58), (4.59), and (4.60) satisfies

$$\limsup_{t\to\infty} \mathbb{E}^{\varepsilon_{i0}}[\|x_i(t) - x_0(t)\|^2] \le \frac{c_0}{c_1\lambda_{\min}(\mathfrak{L}_1\otimes P)}, \quad i = 1,\ldots,N, \tag{4.62}$$

where

$$c_0 \triangleq \sum_{i=1}^{N}\frac{1}{\gamma_1}d_{i1}^2 + \sum_{i=1}^{N}\frac{1}{\gamma_4}d_{i2}^2 + \sum_{i=1}^{N}\frac{\sigma_{d_i}c}{2n_{d_i}}\bar{d}_i^2 + \sum_{i=1}^{N}\frac{c}{2n_{d_i}}\bar{\dot{d}}_i^2 + N\frac{\bar{\delta}^2}{\gamma_3}\bar{x}_0^2 + \sum_{i=1}^{N}\frac{\sigma_{\theta_i}}{2n_{\theta_i}}\bar{\theta}_i^2 + \sum_{i=1}^{N}\frac{1}{2n_{\theta_i}}\bar{\dot{\theta}}_i^2 \tag{4.63}$$

and

$$c_1 \triangleq \min\Bigg\{\sigma_{d_1} - 1 - 2n_{d_1}\gamma_1 c, \ldots, \sigma_{d_N} - 1 - 2n_{d_N}\gamma_1 c, \sigma_{\theta_1} - 1 - 2n_{\theta_1}\gamma_4, \ldots, \sigma_{\theta_N} - 1 - 2n_{\theta_N}\gamma_4, \frac{\lambda_{\min}(\mathfrak{L}_1)\lambda_{\min}(R)}{\lambda_{\max}(\mathfrak{L}_1\otimes P)}\Bigg\}. \tag{4.64}$$

Furthermore, for $i \in \{1,\ldots,N\}$ and $t \ge 0$, the adaptive estimates $\hat{d}_i(t)$ and $\hat{\theta}_i(t)$ are ultimately uniformly bounded in a mean-square sense.

Proof: To show ultimate boundedness of the closed-loop system, consider the Lyapunov-like function given by

$$V\big(\varepsilon,\tilde{d},\tilde{\theta}\big) = \varepsilon^{\mathrm{T}}(\mathfrak{L}_1\otimes P)\varepsilon + \sum_{i=1}^{N}\frac{1}{2n_{\theta_i}}\tilde{\theta}_i^{\mathrm{T}}\tilde{\theta}_i + \sum_{i=1}^{N}\frac{c}{2n_{d_i}}\tilde{d}_i^{\mathrm{T}}\tilde{d}_i, \tag{4.65}$$

where $\varepsilon = [\varepsilon_1^{\mathrm{T}}, \ldots, \varepsilon_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{nN}$, $\tilde{d} = [\tilde{d}_1^{\mathrm{T}}, \ldots, \tilde{d}_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{nN}$, $\tilde{\theta} = [\tilde{\theta}_1^{\mathrm{T}}, \ldots, \tilde{\theta}_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{mN}$, and P satisfies (4.61). Now, the infinitesimal generator $\mathcal{L}V\big(\varepsilon, \tilde{d}, \tilde{\theta}\big)$ of the closed-loop system (4.58), (4.59), and (4.60) is given by

$$\begin{aligned}\mathcal{L}V\big(\varepsilon, \tilde{d}, \tilde{\theta}\big) = {} & 2\varepsilon^{\mathrm{T}}(\mathfrak{L}_1 \otimes PA - c\mathfrak{L}_1^2 \otimes PBB^{\mathrm{T}}P)\varepsilon + 2c\sum_{i=1}^{N} \bar{e}_i^{\mathrm{T}} PBB^{\mathrm{T}}P\tilde{d}_i \\ & - 2\sum_{i=1}^{N} \bar{e}_i^{\mathrm{T}} PB\tilde{\theta}_i + \mathrm{tr}[g\varepsilon^{\mathrm{T}}\left(\mathfrak{L}_1 \otimes P\right)\varepsilon g^{\mathrm{T}}] + 2\sum_{i=1}^{N} \bar{e}_i^{\mathrm{T}} P\Delta A_i \varepsilon_i \\ & + 2\sum_{i=1}^{N} \bar{e}_i^{\mathrm{T}} P\Delta A_i x_0 + \sum_{i=1}^{N} \frac{c}{n_{d_i}} \tilde{d}_i^{\mathrm{T}}(\dot{\hat{d}}_i - \dot{d}_i) + \sum_{i=1}^{N} \frac{1}{n_{\theta_i}} \tilde{\theta}_i^{\mathrm{T}}(\dot{\hat{\theta}}_i - \dot{\theta}_i), \\ & \qquad (\varepsilon, \tilde{d}, \tilde{\theta}) \in \mathbb{R}^{Nn} \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nm}.\end{aligned} \tag{4.66}$$

Next, note that, for every $\gamma_1 > 0$,

$$\begin{aligned}2c\bar{e}_i^{\mathrm{T}} PBB^{\mathrm{T}}P\tilde{d}_i &= 2ce_i^{\mathrm{T}} PBB^{\mathrm{T}}P\tilde{d}_i - 2cd_i^{\mathrm{T}} PBB^{\mathrm{T}}P\tilde{d}_i \\ &\leq 2ce_i^{\mathrm{T}} PBB^{\mathrm{T}}P\tilde{d}_i + \frac{1}{\gamma_1}\|d_i^{\mathrm{T}} PBB^{\mathrm{T}}P\|^2 + \gamma_1 c^2 \tilde{d}_i^{\mathrm{T}}\tilde{d}_i\end{aligned} \tag{4.67}$$

and, using (4.58),

$$\begin{aligned}\frac{c}{n_{d_i}} \tilde{d}_i^{\mathrm{T}}(\dot{\hat{d}}_i - \dot{d}_i) &= -2ce_i^{\mathrm{T}} PBB^{\mathrm{T}}P\tilde{d}_i - \frac{\sigma_{d_i} c}{n_{d_i}} \tilde{d}_i^{\mathrm{T}} \hat{d}_i - \frac{c}{n_{d_i}} \tilde{d}_i^{\mathrm{T}} \dot{d}_i \\ &\leq -2ce_i^{\mathrm{T}} PBB^{\mathrm{T}}P\tilde{d}_i - \frac{(\sigma_{d_i} - 1)c}{2n_{d_i}} \tilde{d}_i^{\mathrm{T}} \tilde{d}_i + \frac{\sigma_{d_i} c}{2n_{d_i}} \bar{d}_i^2 + \frac{c}{2n_{d_i}} \bar{\dot{d}}_i^2.\end{aligned} \tag{4.68}$$

Thus,

$$2c\bar{e}_i^{\mathrm{T}} PBB^{\mathrm{T}}P\tilde{d}_i + \frac{c}{n_{d_i}} \tilde{d}_i^{\mathrm{T}}(\dot{\hat{d}}_i - \dot{d}_i) \leq -\frac{(\sigma_{d_i} - 1 - 2n_{d_i}\gamma_1 c)c}{2n_{d_i}} \tilde{d}_i^{\mathrm{T}} \tilde{d}_i + \frac{\sigma_{d_i} c}{2n_{d_i}} \bar{d}_i^2 + \frac{c}{2n_{d_i}} \bar{\dot{d}}_i^2 + \frac{1}{\gamma_1} d_{i1}^2. \tag{4.69}$$

A similar analysis shows

$$-2\bar{e}_i^{\mathrm{T}} PB\tilde{\theta}_i + \frac{1}{n_{\theta_i}} \tilde{\theta}_i^{\mathrm{T}}(\dot{\hat{\theta}}_i - \dot{\theta}_i) \leq -\frac{\sigma_{\theta_i} - 1 - 2n_{\theta_i}\gamma_4}{2n_{\theta_i}} \tilde{\theta}_i^{\mathrm{T}} \tilde{\theta}_i + \sum_{i=1}^{N} \frac{\sigma_{\theta_i}}{2n_{\theta_i}} \bar{\theta}_i^2 + \sum_{i=1}^{N} \frac{1}{2n_{\theta_i}} \bar{\dot{\theta}}_i^2 + \sum_{i=1}^{N} \frac{1}{\gamma_4} d_{i2}^2. \tag{4.70}$$

Next, for $\gamma_2 > 0$ and $\gamma_3 > 0$, it can be shown that

$$2\sum_{i=1}^{N} \bar{e}_i^{\mathrm{T}} P\Delta A_i \varepsilon_i \leq \sum_{i=1}^{N} \gamma_2 \bar{e}_i^{\mathrm{T}} P^2 \bar{e}_i + \sum_{i=1}^{N} \frac{\bar{\delta}^2}{\gamma_2} \varepsilon_i^{\mathrm{T}} \varepsilon_i$$

$$= \gamma_2 \varepsilon^{\mathrm{T}} (\mathfrak{L}_1 \otimes I_n) \left(I_N \otimes P^2\right) (\mathfrak{L}_1 \otimes I_n) \varepsilon + \frac{\bar{\delta}^2}{\gamma_2} \varepsilon^{\mathrm{T}} \varepsilon$$

$$= \gamma_2 \varepsilon^{\mathrm{T}} (\mathfrak{L}_1^2 \otimes P^2) \varepsilon + \frac{\bar{\delta}^2}{\gamma_2} \varepsilon^{\mathrm{T}} \varepsilon \tag{4.71}$$

and

$$2 \sum_{i=1}^{N} \bar{e}_i^{\mathrm{T}} P \Delta A_i x_0 \leq \sum_{i=1}^{N} \gamma_3 \bar{e}_i^{\mathrm{T}} P^2 \bar{e}_i + N \frac{\bar{\delta}^2}{\gamma_3} x_0^{\mathrm{T}} x_0$$

$$= \gamma_3 \varepsilon^{\mathrm{T}} (\mathfrak{L}_1^2 \otimes P^2) \varepsilon + N \frac{\bar{\delta}^2}{\gamma_3} x_0^{\mathrm{T}} x_0. \tag{4.72}$$

Now, using (4.67)–(4.72) and the fact that $\mathrm{tr}[g \varepsilon^{\mathrm{T}} (\mathfrak{L}_1 \otimes P) \varepsilon g^{\mathrm{T}}] = \|g\|^2 \varepsilon^{\mathrm{T}} [\mathfrak{L}_1 \otimes P] \varepsilon$, it follows from (4.66), after some algebraic manipulation, that

$$\begin{aligned} \mathcal{L}V\big(\varepsilon, \tilde{d}, \tilde{\theta}\big) \leq{}& \varepsilon^{\mathrm{T}} [\mathfrak{L}_1 \otimes (PA + A^{\mathrm{T}} P) - 2c \mathfrak{L}_1^2 \otimes PBB^{\mathrm{T}} P \\ & + (\gamma_2 + \gamma_3) \mathfrak{L}_1^2 \otimes P^2 + \frac{\bar{\delta}^2}{\gamma_2} I_n + \|g\|^2 \mathfrak{L}_1 \otimes P] \varepsilon \\ & - \sum_{i=1}^{N} \frac{(\sigma_{d_i} - 1 - 2n_{d_i} \gamma_1 c) c}{2n_{d_i}} \tilde{d}_i^{\mathrm{T}} \tilde{d}_i - \sum_{i=1}^{N} \frac{\sigma_{\theta_i} - 1 - 2n_{\theta_i} \gamma_4}{2n_{\theta_i}} \tilde{\theta}_i^{\mathrm{T}} \tilde{\theta}_i + c_0, \\ & (\varepsilon, \tilde{d}, \tilde{\theta}) \in \mathbb{R}^{Nn} \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nm}, \end{aligned} \tag{4.73}$$

where c_0 is given by (4.63).

Next, since $\mathfrak{L}_1$ is positive definite, there exists an orthogonal matrix $T \in \mathbb{R}^{N \times N}$ such that $T^{\mathrm{T}} \mathfrak{L}_1 T = \mathrm{diag}[\lambda_1, \ldots, \lambda_N]$, where λ_i, $i \in \{1, \ldots, N\}$, are the eigenvalues of $\mathfrak{L}_1$. Defining $\xi \triangleq (T^{\mathrm{T}} \otimes I_n) \varepsilon$, it follows from (4.73) that

$$\begin{aligned} \mathcal{L}V\big(\varepsilon, \tilde{d}, \tilde{\theta}\big) \leq{}& \sum_{i=1}^{N} \lambda_i \xi_i^{\mathrm{T}} [P \tilde{A} + \tilde{A}^{\mathrm{T}} P - 2c \lambda_{\min}(\mathfrak{L}_1) PBB^{\mathrm{T}} P \\ & + (\gamma_2 + \gamma_3) \lambda_{\max}(\mathfrak{L}_1) P^2 + \frac{\bar{\delta}^2}{\gamma_2 \lambda_i} I_n] \xi_i \\ & - \sum_{i=1}^{N} \frac{(\sigma_{d_i} - 1 - 2n_{d_i} \gamma_1 c) c}{2n_{d_i}} \tilde{d}_i^{\mathrm{T}} \tilde{d}_i - \sum_{i=1}^{N} \frac{\sigma_{\theta_i} - 1 - 2n_{\theta_i} \gamma_4}{2n_{\theta_i}} \tilde{\theta}_i^{\mathrm{T}} \tilde{\theta}_i + c_0 \\ \leq{}& -\lambda_{\min}(\mathfrak{L}_1) \lambda_{\min}(R) \varepsilon^{\mathrm{T}} \varepsilon - \sum_{i=1}^{N} \frac{(\sigma_{d_i} - 1 - 2n_{d_i} \gamma_1 c) c}{2n_{d_i}} \tilde{d}_i^{\mathrm{T}} \tilde{d}_i \\ & - \sum_{i=1}^{N} \frac{\sigma_{\theta_i} - 1 - 2n_{\theta_i} \gamma_4}{2n_{\theta_i}} \tilde{\theta}_i^{\mathrm{T}} \tilde{\theta}_i + c_0, \quad (\varepsilon, \tilde{d}, \tilde{\theta}) \in \mathbb{R}^{Nn} \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nm}. \end{aligned} \tag{4.74}$$

Now, it follows from (4.74) that

$$\mathcal{L}V\big(\varepsilon, \tilde{d}, \tilde{\theta}\big) \leq -c_1 V\big(\varepsilon, \tilde{d}, \tilde{\theta}\big) + c_0, \quad (\varepsilon, \tilde{d}, \tilde{\theta}) \in \mathbb{R}^{Nn} \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nm}, \tag{4.75}$$

where c_1 is given by (4.64).

Next, using Lemma 3.1, it follows from (4.75) that

$$0 \leq \mathbb{E}^{\varepsilon_0}[V\big(\varepsilon(t), \tilde{d}(t), \tilde{\theta}(t)\big)] \leq V\big(\varepsilon(0), \tilde{d}(0), \tilde{\theta}(0)\big)e^{-c_1 t} + \frac{c_0}{c_1}, \quad t \geq 0, \tag{4.76}$$

and hence, all the signals of the closed-loop system are uniformly ultimately bounded in probability in a mean-square sense. Finally, noting that

$$\limsup_{t\to\infty} \mathbb{E}^{\varepsilon_0}[\varepsilon^{\mathrm{T}}(t)\,(\mathfrak{L}_1 \otimes P)\,\varepsilon(t)] \leq \limsup_{t\to\infty} \mathbb{E}^{\varepsilon_0}[V\big(\varepsilon(t), \tilde{d}(t), \tilde{\theta}(t)\big)] \leq \frac{c_0}{c_1}, \tag{4.77}$$

it follows that, for every $i \in \{1, \ldots, N\}$,

$$\limsup_{t\to\infty} \mathbb{E}^{\varepsilon_{i0}}[\|x_i(t) - x_0(t)\|^2] \leq \frac{c_0}{c_1 \lambda_{\min}(\mathfrak{L}_1 \otimes P)}, \tag{4.78}$$

which implies that the pathwise trajectory of the state tracking error for each agent of the closed-loop system associated with the plant dynamics is uniformly ultimately bounded in a mean-square sense. □

Note that n_{d_i} and n_{θ_i}, $i \in \{1, \ldots, N\}$, are design gain parameters used in the adaptive laws (4.58) and (4.59), respectively, and thus, selecting large values of these parameters can introduce transient oscillations in the update law estimates of $\hat{d}_i(t)$, $t \geq 0$, and $\hat{\theta}_i$, $t \geq 0$. This can be remedied by adding a modification term in the update laws to filter out the high frequency content in the control signal while preserving uniform ultimate boundedness in a mean-square sense. This architecture is developed in [155].

4.8. Illustrative Numerical Example

To illustrate the key ideas presented in this subsection, consider the multiagent system representing the controlled lateral dynamics of four follower aircrafts and one leader aircraft

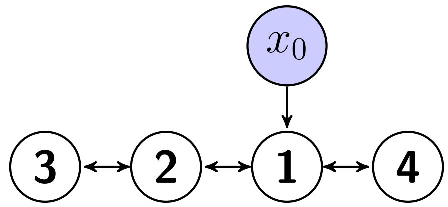


Figure 4.8.1: Leader-follower communication topology $\mathfrak{G}$.

with a communication topology shown in Figure 4.8.1. Node x_0 represents the leader aircraft and nodes 1 through 4 represent the follower aircrafts. For the leader aircraft, the dynamical system representing the lateral directional dynamics of an aircraft [73] are given by (4.50), where, $x_0(t) \triangleq [\beta(t), p(t), r(t)]^{\mathrm{T}}$, $\beta(t)$, $t \geq 0$, is the sideslip angle in deg, $p(t)$, $t \geq 0$, is the roll rate in deg/sec, and $r(t)$, $t \geq 0$, is the yaw rate in deg/sec. Here we take $x_{00} = [1, -2, -1]^{\mathrm{T}}$ and $g = [1,\ 1]^{\mathrm{T}}$. The state-dependent disturbance is used to capture perturbations in atmospheric drag [82]. Furthermore, the system matrices are given by

$$A = \begin{bmatrix} -0.025 & 0.104 & -0.994 \\ 574.7 & 0 & 0 \\ 16.20 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.122 & -0.276 \\ -53.61 & 33.25 \\ 195.5 & -529.4 \end{bmatrix}, \tag{4.79}$$

with reference input

$$r_0(t) = \begin{bmatrix} 2.0738 & 0.0665 & -0.0573 \\ -3.8006 & -0.0950 & 0.1398 \end{bmatrix} x_0(t). \tag{4.80}$$

The follower aircraft dynamics are given by (4.49), where $x_1(t) \triangleq [\beta_1(t), p_1(t), r_1(t)]^{\mathrm{T}}$, $x_{10} = [-2, -3, -1]^{\mathrm{T}}$, $x_2(t) \triangleq [\beta_2(t), p_2(t), r_2(t)]^{\mathrm{T}}$, $x_{20} = [-3, 0, -3]^{\mathrm{T}}$, $x_3(t) \triangleq [\beta_3(t), p_3(t), r_3(t)]^{\mathrm{T}}$, $x_{30} = [-2, -1, -3]^{\mathrm{T}}$, and $x_4(t) \triangleq [\beta_4(t), p_4(t), r_4(t)]^{\mathrm{T}}$, $x_{40} = [-5, -1, -2]^{\mathrm{T}}$. Furthermore, we assume the follower agent model uncertainties

$$\Delta A_1 = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.06 \end{bmatrix}, \quad \Delta A_2 = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.06 & 0 \\ 0 & 0 & 0.04 \end{bmatrix},$$
$$\Delta A_3 = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.06 & 0 \\ 0 & 0 & 0.04 \end{bmatrix}, \quad \Delta A_4 = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.06 \end{bmatrix}. \tag{4.81}$$

The uncorrupted control inputs for the follower agents $u_i(t) \triangleq \left[\delta_{\mathrm{ail},i}(t),\ \delta_{\mathrm{rud},i}(t)\right]^{\mathrm{T}}$, $i \in \{1,2,3,4\}$, involve the aileron command in deg and the rudder command in deg, respectively. The actuator attacks are characterized as

$$\phi_1(t) = \begin{bmatrix} 0.1(1-e^{-0.1t}) \\ 0.08(1-e^{-0.15t}) \end{bmatrix}, \quad \phi_2(t) = \begin{bmatrix} 0.2(1-e^{-0.1t}) \\ 0.1(1-e^{-0.15t}) \end{bmatrix}, \tag{4.82}$$

$$\phi_3(t) = \begin{bmatrix} 0.15(1-e^{-0.1t}) \\ 0.1(1-e^{-0.15t}) \end{bmatrix}, \quad \phi_4(t) = \begin{bmatrix} 0.2(1-e^{-0.1t}) \\ 0.08(1-e^{-0.15t}) \end{bmatrix}. \tag{4.83}$$

Furthermore, the neighbourhood synchronization error attacks are modeled as

$$d_1(t) = 0.1\sin(8t), \quad d_2(t) = 0.2\cos(4t), \quad d_3(t) = 0.2\sin(5t), \quad d_4(t) = 0.1\cos(6t). \tag{4.84}$$

To design a distributed robust adaptive controller we use Theorem 4.2 with

$$P = \begin{bmatrix} 233.9464 & 0.3843 & -0.1411 \\ 0.3843 & 0.1401 & 0.0113 \\ -0.1411 & 0.0113 & 0.0107 \end{bmatrix} \tag{4.85}$$

and design parameters $c = 2$, $\gamma_1 = 1$, $\gamma_2 = 0.1$, $\gamma_3 = 0.1$, $\gamma_4 = 1$, $n_{d_i} = 3$, $i \in \{1,2,3,4\}$, $\sigma_{d_i} = 13.2$, $i \in \{1,2,3,4\}$, $n_{\theta_i} = 3$, $i \in \{1,2,3,4\}$, and $\sigma_{\theta_i} = 7.2$, $i \in \{1,2,3,4\}$. The system performance of the controller given by (4.57) with the proposed robust adaptive scheme is shown in Figures 4.8.2-4.8.5 for the ith follower agent, where $i \in \{1,2,3,4\}$. Specifically, Figures 4.8.2-4.8.5 show a sample trajectory along with the standard deviation of the state tracking error $\varepsilon_i(t) = x_i(t) - x_0(t)$ for agent $i \in \{1,2,3,4\}$ versus time for 10 sample paths. The mean control profile is also plotted in Figures 4.8.2-4.8.5. It follows from Theorem 4.2 that the state tracking error for each agent is guaranteed to be uniformly ultimate bounded in a mean-square sense.

4.9. An Output Feedback Adaptive Controller for Leader-Follower Multiagent Systems with Stochastic Disturbances and Sensor-Actuator Attacks

In this section, we discuss the case where only system output measurements of the agents are available for feedback. Specifically, consider a leader-follower networked mul-

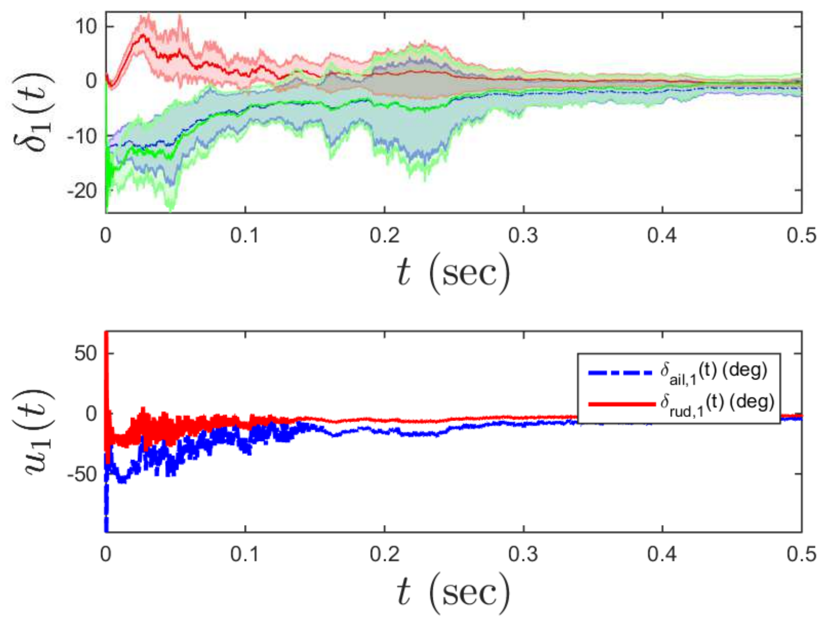


Figure 4.8.2: Agent 1 sample average of the state tracking error profile along with the sample standard deviation of the closed-loop nominal system trajectories versus time; $\beta_1(t) - \beta(t)$ in blue, $p_1(t) - p(t)$ in red, and $r_1(t) - r(t)$ in green. The control profile is plotted as the mean of the 10 sample runs.

tiagent system consisting of N follower agents with dynamics and output measurements of agent $i \in \{1, \ldots, N\}$ given by

$$\mathrm{d}x_i(t) = [Ax_i(t) + Bu_i(t)]\mathrm{d}t + x_i(t)g^{\mathrm{T}}\mathrm{d}w(t), \quad x_i(0) \stackrel{\text{a.s.}}{=} x_{i0}, \quad t \geq 0, \tag{4.86}$$

$$y_i(t) = Cx_i(t), \tag{4.87}$$

where, for $i \in \{1, \ldots, N\}$ and $t \geq 0$, $x_i(t) \in \mathcal{H}_n$ and $y_i(t) \in \mathcal{H}_q$ are the state and output of the ith follower agent, respectively, $u_i(t) \in \mathcal{H}_m$ is the uncorrupted control input to the ith follower agent, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{q\times n}$ are system matrices with (A, B) stabilizable and (A, C) detectable, $w(\cdot)$ is a d-dimensional independent standard Wiener process (i.e., Brownian motion) defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and $g \in \mathbb{R}^d$. Furthermore, we assume that $u_i(t) \in \mathcal{H}_m$ satisfies similar assumptions as in Section 4.6 so that (4.86) has a unique solution forward in time.

As in the state feedback case, we assume that the control input of the ith follower agent

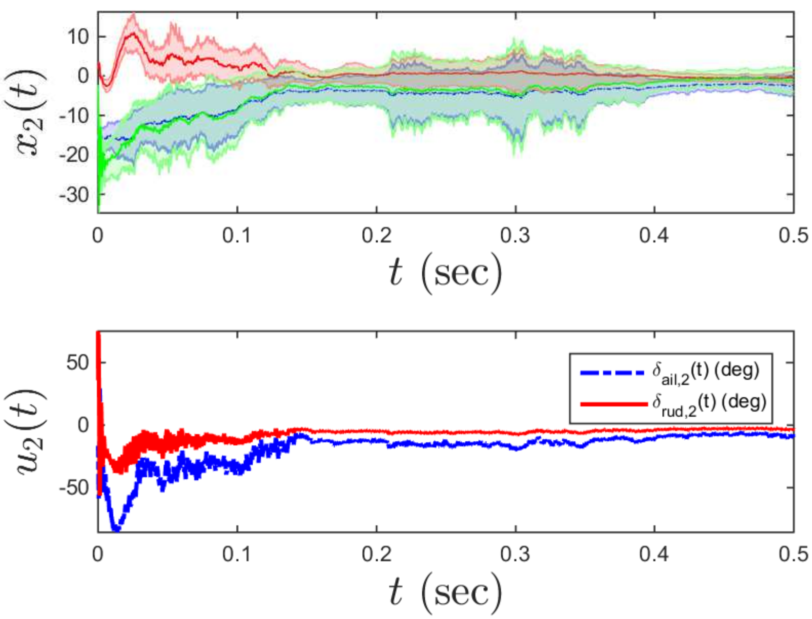


Figure 4.8.3: Agent 2 sample average of the state tracking error profile along with the sample standard deviation of the closed-loop nominal system trajectories versus time; $\beta_2(t) - \beta(t)$ in blue, $p_2(t) - p(t)$ in red, and $r_2(t) - r(t)$ in green. The control profile is plotted as the mean of the 10 sample runs.

is compromised and is given by (4.48), with the compromised controlled system given by

$$\mathrm{d}x_i(t) = [Ax_i(t) + B\tilde{u}_i(t)]\mathrm{d}t + x_i(t)g^{\mathrm{T}}\mathrm{d}w(t), \quad x_i(0) \stackrel{\text{a.s.}}{=} x_{i0}, \quad i = 1, \ldots, N, \quad t \geq 0, \tag{4.88}$$

$$y_i(t) = Cx_i(t). \tag{4.89}$$

Moreover, the leader dynamics are given by

$$\mathrm{d}x_0(t) = [Ax_0(t) + Br_0(t)]\mathrm{d}t + x_0(t)g^{\mathrm{T}}\mathrm{d}w(t), \quad x_0(0) \stackrel{\text{a.s.}}{=} x_{00}, \quad t \geq 0, \tag{4.90}$$

$$y_0(t) = Cx_0(t), \tag{4.91}$$

where $x_0(t) \in \mathcal{H}_n$, $t \geq 0$, and $y_0(t) \in \mathcal{H}_q$, $t \geq 0$, are the leader state and output, respectively, and $r_0(t) \in \mathbb{R}^m$, $t \geq 0$, is a bounded continuous reference input.

Next, it follows from (4.51) that the state neighborhood synchronization error can be written in compact form as

$$\bar{e}_x(t) = (\mathfrak{L}_1 \otimes I_n)\varepsilon(t), \tag{4.92}$$

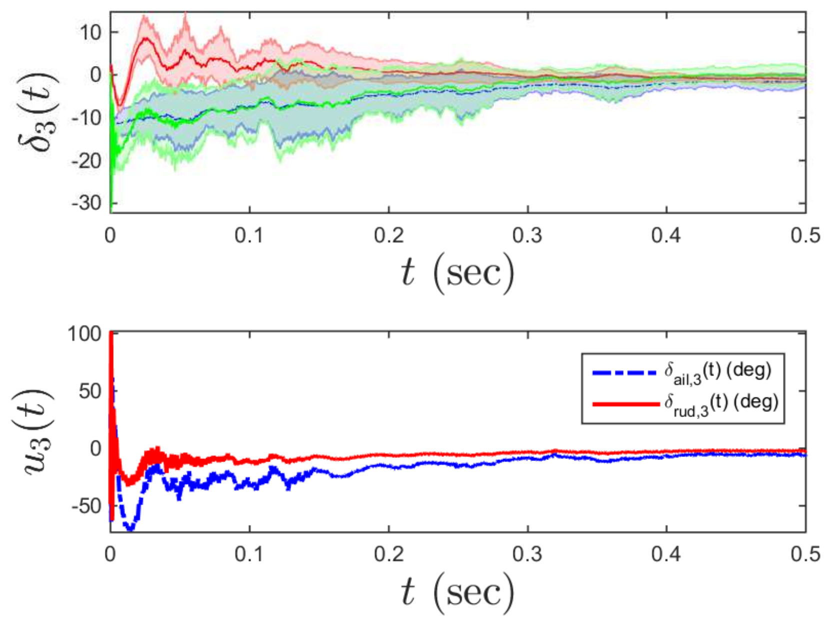


Figure 4.8.4: Agent 3 sample average of the state tracking error profile along with the sample standard deviation of the closed-loop nominal system trajectories versus time; $\beta_3(t)-\beta(t)$ in blue, $p_3(t)-p(t)$ in red, and $r_3(t)-r(t)$ in green. The control profile is plotted as the mean of the 10 sample runs.

where $\bar{e}_x(t) = [\bar{e}_{x1}^{\mathrm{T}}(t), \ldots, \bar{e}_{xN}^{\mathrm{T}}(t)]^{\mathrm{T}}$ and $\varepsilon(t) = [\varepsilon_1^{\mathrm{T}}(t), \ldots, \varepsilon_N^{\mathrm{T}}(t)]^{\mathrm{T}}$. The control objective in this section is to design a distributed output feedback control algorithm such that $\varepsilon(t)$, $t \geq 0$, is uniformly ultimately bounded in a mean-square sense.

As in the state feedback case considered in Section III, the output feedback leader-follower consensus problem formulation typically assumes a relative output information between neighboring agents in order to derive the ith agent controller [131, 132, 142, 143]. Specifically, for $i \in \{1, \ldots, N\}$, the output neighborhood synchronization error is given by

$$\bar{e}_{yi}(t) = \sum_{j=0}^{N} \mathcal{A}_{(i,j)}[y_i(t) - y_j(t)], \tag{4.93}$$

which assumes exact measurements of the leader information $y_0(t)$, $t \geq 0$, by the ith follower agent, as well as exact output measurements of all neighboring follower agents $y_j(t)$, $t \geq 0$, by the ith follower agent for every $j \in \mathcal{N}_{\text{in}}(i)$. As noted earlier, in the case where we have communication channel attacks or when the sensors measuring the leader output as well as

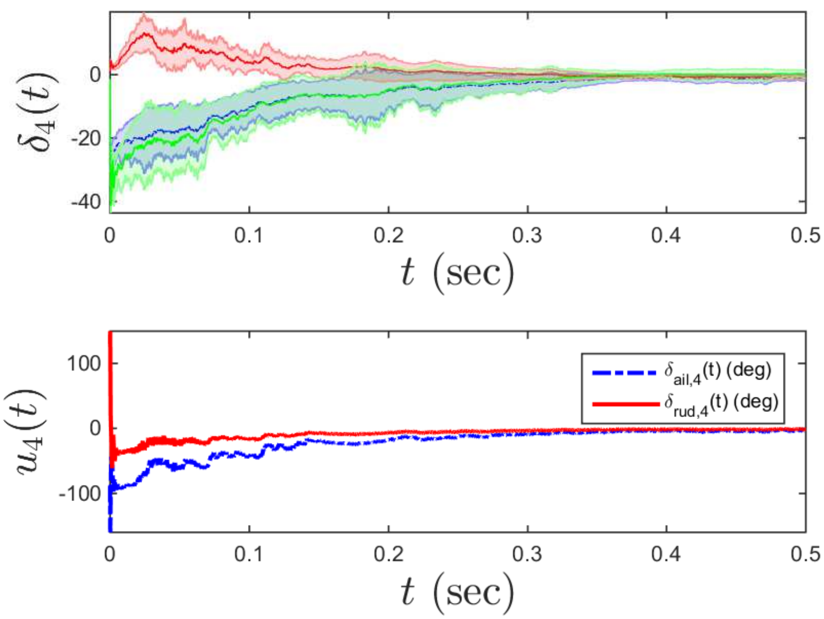


Figure 4.8.5: Agent 4 sample average of the state tracking error profile along with the sample standard deviation of the closed-loop nominal system trajectories versus time; $\beta_4(t)-\beta(t)$ in blue, $p_4(t)-p(t)$ in red, and $r_4(t)-r(t)$ in green. The control profile is plotted as the mean of the 10 sample runs.

the neighboring follower outputs are under attack, the output neighborhood synchronization error may not be accurately available to the agents. To address uncertainty in the output neighborhood synchronization error we use the output neighborhood synchronization error given by

$$e_{yi}(t) = \bar{e}_{yi}(t) + d_{yi}(t), \tag{4.94}$$

where $d_{yi}(t)$, $i \in \{1,\ldots,N\}$ and $t \geq 0$, quantifies information uncertainty between the follower agents as well as the leader agent.

The following assumption is necessary for the main results of Section 4.10.

Assumption 4.4. For $i \in \{1,\ldots,N\}$ and $t \geq 0$, there exist *unknown* scalars $\bar{d}_{yi}$, $\bar{\dot{d}}_{yi}$, $\bar{\phi}_i$, $\bar{\dot{\phi}}_i$, and $\bar{r}_0$ such that $\|\dot{d}_{yi}(t)\| \leq \bar{\dot{d}}_{yi}$, $\|d_{yi}(t)\| \leq \bar{d}_{yi}$, $\|\dot{\phi}_i(t)\| \leq \bar{\dot{\phi}}_i$, $\|\phi_i(t)\| \leq \bar{\phi}_i$, and $\|r_0(t)\| \leq \bar{r}_0$.

4.10. Distributed Adaptive Output Feedback Control Design

In this section, we develop a distributed output feedback adaptive control architecture for the stochastic multiagent system given by (4.88), (4.89), (4.90), and (4.91). The control action for the ith follower agent is given by

$$u_i(t) = cK\vartheta_i(t) - \hat{\phi}_i(t), \tag{4.95}$$

where

$$\begin{aligned}\dot{\vartheta}_i(t) &= (A + LC)\vartheta_i(t) + cBK\left[\sum_{j=1}^{N}\mathcal{A}_{(i,j)}[\vartheta_i(t) - \vartheta_j(t)] + \mathcal{A}_{(i,0)}\vartheta_i(t)\right] - Le_{yi}(t) \\ &\quad - B\left[\sum_{j=1}^{N}\mathcal{A}_{(i,j)}[\hat{\phi}_i(t) - \hat{\phi}_j(t)] + \mathcal{A}_{(i,0)}\hat{\phi}_i(t)\right] + L\hat{d}_{yi}(t), \quad \vartheta_i(0) = \vartheta_{i0}, \quad t \geq 0, \end{aligned} \tag{4.96}$$

$c > 0$ is a design constant, $K \in \mathbb{R}^{m\times n}$ is a gain matrix to be determined, $L \in \mathbb{R}^{n\times q}$ is a filter gain matrix such that $\tilde{A}_\gamma + LC$ is Hurwitz, where $\tilde{A}_\gamma \triangleq A + (\frac{1}{2}\|g\|^2 + \gamma)I_n$ and $\gamma > 0$, and, for $i \in \{1, \ldots, N\}$ and $t \geq 0$, $\hat{d}_{yi}(t)$ and $\hat{\phi}_i(t)$ are the estimates of $d_{yi}(t)$ and $\phi_i(t)$, respectively. Recall that (A, C) is detectable if and only if $(\tilde{A}_\gamma, C)$ is detectable.

For $i \in \{1, \ldots, N\}$ the update laws $\hat{d}_{yi}(t) \in \mathcal{H}_q$, $t \geq 0$, and $\hat{\phi}_i(t) \in \mathcal{H}_m$, $t \geq 0$, are given by

$$\mathrm{d}\hat{d}_{yi}(t) = -\left[n_{d_{yi}}(PL)^{\mathrm{T}}\vartheta_i(t) + n_{d_{yi}}\frac{\tau}{\gamma}L^{\mathrm{T}}QL\hat{d}_{yi}(t) + \sigma_{d_{yi}}\hat{d}_{yi}(t)\right]\mathrm{d}t, \quad \hat{d}_{yi}(0) \overset{\text{a.s.}}{=} \hat{d}_{yi0}, \quad t \geq 0, \tag{4.97}$$

$$\mathrm{d}\hat{\phi}_i(t) = \left[n_{\phi_i}(PB)^{\mathrm{T}}\left(\sum_{j=1}^{N}\mathfrak{L}_{(i,j)}\vartheta_j(t)\right) - \sigma_{\phi_i}\hat{\phi}_i(t)\right]\mathrm{d}t, \quad \hat{\phi}_i(0) \overset{\text{a.s.}}{=} \hat{\phi}_{i0}, \tag{4.98}$$

where $P \in \mathbb{R}^{n\times n}$ is a positive-definite solution to

$$\tilde{A}_\gamma^{\mathrm{T}}P + P\tilde{A}_\gamma - 2c\lambda_{\min}(\mathfrak{L}_1)PBB^{\mathrm{T}}P + R = 0 \tag{4.99}$$

for a given positive-definite matrix $R \in \mathbb{R}^{n\times n}$, $\tau > 0$, $n_{d_{yi}} > 0$, $n_{\phi_i} > 0$, $\sigma_{d_{yi}} > 2 + \beta$, and $\sigma_{\phi_i} > 1$ are design constants with $\gamma > 0$, $\beta > 0$, and $c > 0$, and $Q \in \mathbb{R}^{n\times n}$ is a positive-definite solution to

$$(\tilde{A}_\gamma + LC)^{\mathrm{T}}Q + Q(\tilde{A}_\gamma + LC) < 0. \tag{4.100}$$

Next, using (4.95) the dynamics for the state neighborhood synchronization error $\bar{e}_{xi}(t)$, $i \in \{1, \ldots, N\}$, $t \geq 0$, are given by

$$\begin{aligned}
\mathrm{d}\bar{e}_{xi}(t) = &\Bigg(A\bar{e}_{xi}(t) + cBK\Bigg[\sum_{j=1}^{N} \mathcal{A}_{(i,j)}(\vartheta_i(t) - \vartheta_j(t)) + \mathcal{A}_{(i,0)}\vartheta_i(t)\Bigg] \\
&- B\Bigg[\sum_{j=1}^{N} \mathcal{A}_{(i,j)}(\tilde{\phi}_i(t) - \tilde{\phi}_j(t)) + \mathcal{A}_{(i,0)}\tilde{\phi}_i(t)\Bigg] - \mathcal{A}_{(i,0)}Br_0(t)\Bigg)\mathrm{d}t \\
&+ \bar{e}_{xi}(t)g^{\mathrm{T}}\mathrm{d}w(t), \quad \bar{e}_{xi}(0) \overset{\text{a.s.}}{=} \bar{e}_{xi0}, \quad t \geq 0,
\end{aligned} \tag{4.101}$$

where $\tilde{\phi}_j(t) \triangleq \hat{\phi}_j(t) - \phi_j(t)$. Now, for $i \in \{1, \ldots, N\}$, defining $z_i(t) \triangleq [\bar{e}_{xi}^{\mathrm{T}}(t),\ \vartheta_i^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{2n}$, it follows that

$$\begin{aligned}
\mathrm{d}z_i(t) = &\Bigg(Mz_i(t) + c\sum_{j=1}^{N} \mathfrak{L}_{(i,j)}Hz_j(t) - \eta_i(t) - \sum_{j=1}^{N} \mathfrak{L}_{(i,j)} \begin{bmatrix} B \\ B \end{bmatrix} \tilde{\phi}_j(t) + \begin{bmatrix} 0 \\ L \end{bmatrix} \tilde{d}_{yi}(t)\Bigg)\mathrm{d}t \\
&+ z_i(t) \begin{bmatrix} g \\ 0 \end{bmatrix} \mathrm{d}w(t), \quad z_i(0) \overset{\text{a.s.}}{=} z_{i0}, \quad t \geq 0,
\end{aligned} \tag{4.102}$$

where $\tilde{d}_{yj}(t) \triangleq \hat{d}_{yj}(t) - d_{yj}(t)$ and

$$\begin{aligned}
M &\triangleq \begin{bmatrix} A & 0 \\ -LC & A + LC \end{bmatrix}, \quad H \triangleq \begin{bmatrix} 0 & BK \\ 0 & BK \end{bmatrix}, \\
\eta_i(t) &\triangleq \mathcal{A}_{(i,0)} \begin{bmatrix} Br_0(t) \\ 0 \end{bmatrix} + \sum_{j=1}^{N} \mathfrak{L}_{(i,j)} \begin{bmatrix} 0 \\ B\phi_j(t) \end{bmatrix}.
\end{aligned} \tag{4.103}$$

Theorem 4.3. Consider the stochastic multiagent system given by (4.88), (4.89), (4.90), and (4.91) with actuator attack given by (4.48) and output neighborhood synchronization error given by (4.94). Assume Assumptions 4.1 and 4.4 hold, and (A, B) is stabilizable and (A, R) is observable for a given positive-definite matrix $R \in \mathbb{R}^{n \times n}$. Then there exist positive-definite matrices $P \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{n \times n}$ such that (4.99) and (4.100) hold. Furthermore, with the controller given by (4.95), adaptive laws given by (4.97) and (4.98), and control gain $K = -B^{\mathrm{T}}P$, the closed-loop system given by (4.97), (4.98), and (4.102) satisfies

$$\limsup_{t\to\infty} \mathbb{E}^{\varepsilon_0}[\|\varepsilon(t)\|^2] \le \frac{c_0}{c_1\lambda_{\min}(\tilde{P})\lambda_{\min}(\mathfrak{L}_1)}, \tag{4.104}$$

where

$$\tilde{P} = \begin{bmatrix} \tau Q & -\tau Q \\ -\tau Q & P + \tau Q \end{bmatrix} > 0, \quad \tau > 0, \tag{4.105}$$

$$c_0 \triangleq \sum_{i=1}^{N} \frac{\tau^2}{\gamma^2}\frac{n_{d_{yi}}}{4\beta} d_{yi1} + \sum_{i=1}^{N}\frac{1}{\gamma} d_{yi2} + \sum_{i=1}^{N}\frac{\sigma_{d_{yi}}}{2n_{d_{yi}}}\bar{d}_{yi}^2 + \sum_{i=1}^{N}\frac{1}{2n_{d_{yi}}}\bar{\dot{d}}_{yi}^2 + \sum_{i=1}^{N}\frac{\sigma_{\phi_i}}{2n_{\phi_i}}\bar{\phi}_i^2 + \sum_{i=1}^{N}\frac{1}{2n_{\phi_i}}\bar{\dot{\phi}}_i^2,$$

with d_{yi1} and d_{yi2} satisfying $\|L^{\mathrm{T}}QLd_{yi}(t)\|^2 \le d_{yi1}$, $t \ge 0$, and $\|\eta_i(t)^{\mathrm{T}}\tilde{P}\eta_i(t)\| \le d_{yi2}$, $t \ge 0$, for $i \in \{1, \ldots, N\}$, and

$$c_1 \triangleq \min\left\{\sigma_{d_1} - 2 - \beta, \ldots, \sigma_{d_N} - 2 - \beta, \sigma_{\phi_1} - 1, \ldots, \sigma_{\phi_N} - 1, \frac{\lambda_{\min}(\Omega)\sigma_{\min}(T^{\mathrm{T}}T)}{\lambda_{\max}(\tilde{P})}\right\},$$

where

$$\Omega \triangleq -\begin{bmatrix} \tau[Q(\tilde{A}_\gamma + LC) + (\tilde{A}_\gamma + LC)^{\mathrm{T}}Q] & -C^{\mathrm{T}}L^{\mathrm{T}}P \\ -PLC & -R \end{bmatrix}, \quad T = \begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix}. \tag{4.106}$$

Finally, for $i \in \{1, \ldots, N\}$ and $t \ge 0$, the adaptive estimates $\hat{d}_{yi}(t)$ and $\hat{\phi}_i(t)$ are ultimately uniformly bounded in a mean-square sense.

Proof: First, note that (A, B) (resp., (A, R)) is stabilizable (resp., observable) if and only if $(\tilde{A}_\gamma, B)$ (resp., $(\tilde{A}_\gamma, R)$) is stabilizable (resp., observable). Now, it follows from standard Riccati equation theory [46] that there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that (4.99) holds. In addition, since $\tilde{A}_\gamma + LC$ is Hurwitz, it follows from converse Lyapunov theory [51] that there exists a positive definite $Q \in \mathbb{R}^{n\times n}$ such that (4.100) holds. And since P and Q are positive definite, there exists $\tau > 0$ such that (4.105) holds.

To show ultimate boundedness of the closed-loop system, consider the Lyapunov-like function given by

$$V\big(z, \tilde{d}_y, \tilde{\phi}\big) = z^{\mathrm{T}}(I_N \otimes \tilde{P})z + \sum_{i=1}^{N}\frac{1}{n_{\phi_i}}\tilde{\phi}_i^{\mathrm{T}}\tilde{\phi}_i + \sum_{i=1}^{N}\frac{1}{n_{d_{yi}}}\tilde{d}_{yi}^{\mathrm{T}}\tilde{d}_{yi}, \tag{4.107}$$

where $z=[z_1^{\mathrm{T}},\ldots,z_N^{\mathrm{T}}]^{\mathrm{T}}\in\mathbb{R}^{2nN}$, $\tilde{d}_y=[\tilde{d}_{y1}^{\mathrm{T}},\ldots,\tilde{d}_{yN}^{\mathrm{T}}]^{\mathrm{T}}\in\mathbb{R}^{qN}$, $\tilde{\phi}=[\tilde{\phi}_1^{\mathrm{T}},\ldots,\tilde{\phi}_N^{\mathrm{T}}]^{\mathrm{T}}\in\mathbb{R}^{mN}$, and $\tilde{P}$ given by (4.105). Now, the infinitesimal generator $\mathcal{L}V\big(z,\tilde{d}_y,\tilde{\phi}\big)$ of the closed-loop system (4.97), (4.98), and (4.102) is given by

$$\begin{aligned}\mathcal{L}V\big(z,\tilde{d}_y,\tilde{\phi}\big)&=2z^{\mathrm{T}}(I_N\otimes\tilde{P})(I_N\otimes M+c\mathfrak{L}_1\otimes H)z-2z^{\mathrm{T}}\left(\mathfrak{L}_1\otimes\tilde{P}\begin{bmatrix}B\\B\end{bmatrix}\right)\tilde{\phi}\\&\quad+\sum_{i=1}^{N}2z_i^{\mathrm{T}}\tilde{P}\begin{bmatrix}0\\L\end{bmatrix}\tilde{d}_{yi}-\sum_{i=1}^{N}2z_i^{\mathrm{T}}\tilde{P}\eta_i+\sum_{i=1}^{N}\frac{2}{n_{d_{yi}}}\tilde{d}_{yi}^{\mathrm{T}}(\dot{\hat{d}}_{yi}-\dot{d}_{yi})\\&\quad+\sum_{i=1}^{N}\frac{2}{n_{\phi_i}}\tilde{\phi}_i^{\mathrm{T}}(\dot{\hat{\phi}}_i-\dot{\phi}_i)+\|g\|^2(z^{\mathrm{T}}[I_N\otimes\tilde{P}]z),\\&\qquad(z,\tilde{d}_y,\tilde{\phi})\in\mathbb{R}^{2Nn}\times\mathbb{R}^{Nq}\times\mathbb{R}^{Nm}.\end{aligned}\tag{4.108}$$

Next, since $\tilde{P}\begin{bmatrix}B\\B\end{bmatrix}=\begin{bmatrix}0\\PB\end{bmatrix}$, the second term in (4.108) can be written as

$$-2z^{\mathrm{T}}\left(\mathfrak{L}_1\otimes\tilde{P}\begin{bmatrix}B\\B\end{bmatrix}\right)\tilde{\phi}=-2\vartheta^{\mathrm{T}}(\mathfrak{L}_1\otimes PB)\tilde{\phi}=-2\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\mathfrak{L}_{(i,j)}\vartheta_j^{\mathrm{T}}\right)PB\tilde{\phi}_i.\tag{4.109}$$

In addition, the last term in (4.108) satisfies

$$\begin{aligned}\sum_{i=1}^{N}\frac{2}{n_{\phi_i}}\tilde{\phi}_i^{\mathrm{T}}(\dot{\hat{\phi}}_i-\dot{\phi}_i)&=\sum_{i=1}^{N}2\tilde{\phi}_i^{\mathrm{T}}(PB)^{\mathrm{T}}\left(\sum_{j=1}^{N}\mathfrak{L}_{(i,j)}\vartheta_j\right)-\sum_{i=1}^{N}\frac{2\sigma_{\phi_i}}{n_{\phi_i}}\tilde{\phi}_i^{\mathrm{T}}\hat{\phi}_i-\sum_{i=1}^{N}\frac{2}{n_{\phi_i}}\tilde{\phi}_i^{\mathrm{T}}\dot{\phi}_i\\&\le\sum_{i=1}^{N}2\tilde{\phi}_i^{\mathrm{T}}(PB)^{\mathrm{T}}\left(\sum_{j=1}^{N}\mathfrak{L}_{(i,j)}\vartheta_j\right)-\sum_{i=1}^{N}\frac{\sigma_{\phi_i}-1}{n_{\phi_i}}\tilde{\phi}_i^{\mathrm{T}}\tilde{\phi}_i\\&\quad+\sum_{i=1}^{N}\frac{\sigma_{\phi_i}}{n_{\phi_i}}\bar{\phi}_i^2+\sum_{i=1}^{N}\frac{1}{n_{\phi_i}}\bar{\dot{\phi}}_i^2.\end{aligned}\tag{4.110}$$

Now, it follows from (4.109) and (4.110) that

$$\begin{aligned}&-2z^{\mathrm{T}}\left(\mathfrak{L}_1\otimes\tilde{P}\begin{bmatrix}B\\B\end{bmatrix}\right)\tilde{\phi}+\sum_{i=1}^{N}\frac{2}{n_{\phi_i}}\tilde{\phi}_i^{\mathrm{T}}(\dot{\hat{\phi}}_i-\dot{\phi}_i)\\&\qquad\le-\sum_{i=1}^{N}\frac{\sigma_{\phi_i}-1}{n_{\phi_i}}\tilde{\phi}_i^{\mathrm{T}}\tilde{\phi}_i+\sum_{i=1}^{N}\frac{\sigma_{\phi_i}}{n_{\phi_i}}\bar{\phi}_i^2+\sum_{i=1}^{N}\frac{1}{n_{\phi_i}}\bar{\dot{\phi}}_i^2.\end{aligned}\tag{4.111}$$

Furthermore, noting $\begin{bmatrix}L\\0\end{bmatrix}^{\mathrm{T}}\tilde{P}\begin{bmatrix}L\\0\end{bmatrix}=\tau L^{\mathrm{T}}QL$, the third term in (4.108) gives

$$\sum_{i=1}^{N}2z_i^{\mathrm{T}}\tilde{P}\begin{bmatrix}0\\L\end{bmatrix}\tilde{d}_{yi}=\sum_{i=1}^{N}2z_i^{\mathrm{T}}\tilde{P}\begin{bmatrix}L\\L\end{bmatrix}\tilde{d}_{yi}-\sum_{i=1}^{N}2z_i^{\mathrm{T}}\tilde{P}\begin{bmatrix}L\\0\end{bmatrix}\tilde{d}_{yi}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} 2\vartheta_i^{\mathrm{T}} PL\tilde{d}_{yi} - \sum_{i=1}^{N} 2z_i^{\mathrm{T}} \tilde{P} \left[\begin{array}{c} L \\ 0 \end{array} \right] \tilde{d}_{yi} \\
&\leq \sum_{i=1}^{N} 2\vartheta_i^{\mathrm{T}} PL\tilde{d}_{yi} + \sum_{i=1}^{N} \gamma z_i^{\mathrm{T}} \tilde{P} z_i + \sum_{i=1}^{N} \frac{\tau}{\gamma} \tilde{d}_{yi}^{\mathrm{T}} L^{\mathrm{T}} QL\tilde{d}_{yi},
\end{aligned} \tag{4.112}
$$

for $\gamma > 0$. Moreover, the fifth term in (4.108) satisfies

$$
\begin{aligned}
\sum_{i=1}^{N} \frac{2}{n_{d_{yi}}} \tilde{d}_{yi}^{\mathrm{T}} (\dot{\hat{d}}_{yi} - \dot{d}_{yi}) &= - \sum_{i=1}^{N} 2\tilde{d}_{yi}^{\mathrm{T}} (PL)^{\mathrm{T}} \vartheta_i - \sum_{i=1}^{N} \frac{\tau}{\gamma} \tilde{d}_{yi}^{\mathrm{T}} L^{\mathrm{T}} QL\hat{d}_{yi} \\
&\quad - \sum_{i=1}^{N} \frac{2\sigma_{d_{yi}}}{n_{d_{yi}}} \tilde{d}_{yi}^{\mathrm{T}} \hat{d}_{yi} - \sum_{i=1}^{N} \frac{2}{n_{d_{yi}}} \tilde{d}_{yi}^{\mathrm{T}} \dot{d}_{yi} \\
&\leq - \sum_{i=1}^{N} 2\tilde{d}_{yi}^{\mathrm{T}} (PL)^{\mathrm{T}} \vartheta_i - \sum_{i=1}^{N} \frac{\tau}{\gamma} \tilde{d}_{yi}^{\mathrm{T}} L^{\mathrm{T}} QL\hat{d}_{yi} \\
&\quad - \sum_{i=1}^{N} \frac{\sigma_{d_{yi}} - 1}{n_{d_{yi}}} \tilde{d}_{yi}^{\mathrm{T}} \tilde{d}_{yi} + \sum_{i=1}^{N} \frac{\sigma_{d_{yi}}}{n_{d_{yi}}} \bar{d}_{yi}^2 + \sum_{i=1}^{N} \frac{1}{n_{d_{yi}}} \bar{\dot{d}}_{yi}^2.
\end{aligned} \tag{4.113}
$$

Now, it follows from (4.112) and (4.113) that

$$
\begin{aligned}
&\sum_{i=1}^{N} 2z_i^{\mathrm{T}} \tilde{P} \left[\begin{array}{c} 0 \\ L \end{array} \right] \tilde{d}_{yi} + \sum_{i=1}^{N} \frac{2}{n_{d_{yi}}} \tilde{d}_{yi}^{\mathrm{T}} (\dot{\hat{d}}_{yi} - \dot{d}_{yi}) \\
&\qquad \leq \sum_{i=1}^{N} \gamma z_i^{\mathrm{T}} \tilde{P} z_i - \sum_{i=1}^{N} \frac{\tau}{\gamma} \tilde{d}_{yi}^{\mathrm{T}} L^{\mathrm{T}} QL d_{yi} \\
&\qquad\quad - \sum_{i=1}^{N} \frac{\sigma_{d_{yi}} - 1}{n_{d_{yi}}} \tilde{d}_{yi}^{\mathrm{T}} \tilde{d}_{yi} + \sum_{i=1}^{N} \frac{\sigma_{d_{yi}}}{n_{d_{yi}}} \bar{d}_{yi}^2 + \sum_{i=1}^{N} \frac{1}{n_{d_{yi}}} \bar{\dot{d}}_{yi}^2.
\end{aligned} \tag{4.114}
$$

Next, note that, for $\beta > 0$,

$$
\begin{aligned}
- \sum_{i=1}^{N} \frac{\tau}{\gamma} \tilde{d}_{yi}^{\mathrm{T}} L^{\mathrm{T}} QL d_{yi} &\leq \sum_{i=1}^{N} \frac{\beta}{n_{d_{yi}}} \tilde{d}_{yi}^{\mathrm{T}} \tilde{d}_{yi} + \sum_{i=1}^{N} \frac{\tau^2}{\gamma^2} \frac{n_{d_{yi}}}{4\beta} d_{yi}^{\mathrm{T}} (L^{\mathrm{T}} QL)^2 d_{yi} \\
&\leq \sum_{i=1}^{N} \frac{\beta}{n_{d_{yi}}} \tilde{d}_{yi}^{\mathrm{T}} \tilde{d}_{yi} + \sum_{i=1}^{N} \frac{\tau^2}{\gamma^2} \frac{n_{d_{yi}}}{4\beta} d_{yi1},
\end{aligned} \tag{4.115}
$$

and hence, (4.114) implies

$$
\begin{aligned}
&\sum_{i=1}^{N} 2z_i^{\mathrm{T}} \tilde{P} \left[\begin{array}{c} 0 \\ L \end{array} \right] \tilde{d}_{yi} + \sum_{i=1}^{N} \frac{2}{n_{d_{yi}}} \tilde{d}_{yi}^{\mathrm{T}} (\dot{\hat{d}}_{yi} - \dot{d}_{yi}) \\
&\leq \sum_{i=1}^{N} \gamma z_i^{\mathrm{T}} \tilde{P} z_i - \sum_{i=1}^{N} \frac{\sigma_{d_{yi}} - 1 - \beta}{n_{d_{yi}}} \tilde{d}_{yi}^{\mathrm{T}} \tilde{d}_{yi} + \sum_{i=1}^{N} \frac{\sigma_{d_{yi}}}{n_{d_{yi}}} \bar{d}_{yi}^2 + \sum_{i=1}^{N} \frac{1}{n_{d_{yi}}} \bar{\dot{d}}_{yi}^2 + \sum_{i=1}^{N} \frac{\tau^2}{\gamma^2} \frac{n_{d_{yi}}}{4\beta} d_{yi1}.
\end{aligned} \tag{4.116}
$$

Finally, for $\gamma > 0$, the fourth term in (4.108) satisfies

$$-\sum_{i=1}^{N} 2z_i^{\mathrm{T}}\tilde{P}\eta_i \leq \sum_{i=1}^{N} \gamma z_i^{\mathrm{T}}\tilde{P}z_i + \sum_{i=1}^{N}\frac{1}{\gamma}\eta_i^{\mathrm{T}}\tilde{P}\eta_i \leq \sum_{i=1}^{N} \gamma z_i^{\mathrm{T}}\tilde{P}z_i + \sum_{i=1}^{N}\frac{1}{\gamma}d_{yi2}. \tag{4.117}$$

Next, letting $\tilde{z} = (I_N \otimes T)z$, it can be shown, after some algebraic manipulation, that (4.108) yields

$$\begin{aligned}\mathcal{L}V\big(z, \tilde{d}_y, \tilde{\phi}\big) \leq{}& 2\tilde{z}^{\mathrm{T}}\left[I_N \otimes \hat{P}\left(\tilde{M} + \Big(\frac{1}{2}\|g\|^2 + \gamma\Big)I_{2n}\right) + c\mathfrak{L}_1 \otimes \hat{P}\tilde{H}\right]\tilde{z}\\ &-\sum_{i=1}^{N}\frac{\sigma_{d_{yi}} - 1 - \beta}{n_{d_{yi}}}\tilde{d}_{yi}^{\mathrm{T}}\tilde{d}_{yi} - \sum_{i=1}^{N}\frac{\sigma_{d_{yi}} - 1}{n_{d_{yi}}}\tilde{d}_{yi}^{\mathrm{T}}\tilde{d}_{yi} + c_0,\\ &(z, \tilde{d}_y, \tilde{\phi}) \in \mathbb{R}^{2Nn} \times \mathbb{R}^{Nq} \times \mathbb{R}^{Nm},\end{aligned} \tag{4.118}$$

where $\hat{P} = T^{-\mathrm{T}}\tilde{P}T^{-1}$, $\tilde{H} = THT^{-1}$, and $\tilde{M} = TMT^{-1}$. Now, since $\mathfrak{L}_1$ is positive definite, there exists an orthogonal matrix $U \in \mathbb{R}^{N\times N}$ such that $U^{\mathrm{T}}\mathfrak{L}_1 U = \mathrm{diag}[\lambda_1, \ldots, \lambda_N]$, where λ_i, $i = 1, \ldots, N$, are the eigenvalues of $\mathfrak{L}_1$. Next, defining $\bar{z} \triangleq (U^{\mathrm{T}} \otimes I_n)\tilde{z}$ it follows from (4.118) that

$$\begin{aligned}\mathcal{L}V\big(z, \tilde{d}_y, \tilde{\phi}\big) \leq{}& \sum_{i=1}^{N}\bar{z}_i^{\mathrm{T}}(\hat{P}\tilde{M}_2 + \tilde{M}_2^{\mathrm{T}}\hat{P} + 2\lambda_i c\hat{P}\tilde{H})\bar{z} - \sum_{i=1}^{N}\frac{\sigma_{d_{yi}} - 1 - \beta}{n_{d_{yi}}}\tilde{d}_{yi}^{\mathrm{T}}\tilde{d}_{yi}\\ &-\sum_{i=1}^{N}\frac{\sigma_{d_{yi}} - 1}{n_{d_{yi}}}\tilde{d}_{yi}^{\mathrm{T}}\tilde{d}_{yi} + c_0, \quad (z, \tilde{d}_y, \tilde{\phi}) \in \mathbb{R}^{2Nn} \times \mathbb{R}^{Nq} \times \mathbb{R}^{Nm},\end{aligned} \tag{4.119}$$

where $\tilde{M}_\gamma = \tilde{M} + \Big(\frac{1}{2}\|g\|^2 + \gamma\Big)I_{2n}$. Moreover, noting that

$$\hat{P}\tilde{M}_\gamma + \tilde{M}_\gamma^{\mathrm{T}}\hat{P} + 2\lambda_i c\hat{P}\tilde{H} = \begin{bmatrix} \tau[Q(\tilde{A}_\gamma + LC) + (\tilde{A}_\gamma + LC)^{\mathrm{T}}Q] & -C^{\mathrm{T}}L^{\mathrm{T}}P \\ -PLC & -R \end{bmatrix} = -\Omega, \tag{4.120}$$

it follows from (4.119) that

$$\begin{aligned}\mathcal{L}V\big(z, \tilde{d}_y, \tilde{\phi}\big) &\leq -\lambda_{\min}(\Omega)\sigma_{\min}(T^{\mathrm{T}}T)z^{\mathrm{T}}z - \sum_{i=1}^{N}\frac{\sigma_{d_{yi}} - 1 - \beta}{n_{d_{yi}}}\tilde{d}_{yi}^{\mathrm{T}}\tilde{d}_{yi} - \sum_{i=1}^{N}\frac{\sigma_{d_{yi}} - 1}{n_{d_{yi}}}\tilde{d}_{yi}^{\mathrm{T}}\tilde{d}_{yi} + c_0\\ &\leq -c_1 V\big(z, \tilde{d}_y, \tilde{\phi}\big) + c_0, \quad (z, \tilde{d}_y, \tilde{\phi}) \in \mathbb{R}^{2Nn} \times \mathbb{R}^{Nq} \times \mathbb{R}^{Nm}.\end{aligned} \tag{4.121}$$

Next, using Lemma 3.1, it follows from (4.121) that

$$0 \leq \mathbb{E}^{z_0}[V\big(z(t), \tilde{d}_y(t), \tilde{\phi}(t)\big)] \leq V\big(z(0), \tilde{d}_y(0), \tilde{\phi}(0)\big)e^{-c_1 t} + \frac{c_0}{c_1}, \quad t \geq 0, \tag{4.122}$$

and hence, all the signals of the closed-loop system are uniformly ultimately bounded in probability in a mean-square sense. Finally, noting that

$$\limsup_{t\to\infty} \mathbb{E}^{z_0}[z^{\mathrm{T}}(t)(I_N \otimes \tilde{P})z(t)] \leq \limsup_{t\to\infty} \mathbb{E}^{z_0}[V\big(z(t), \tilde{d}_y(t), \tilde{\phi}(t)\big)] \leq \frac{c_0}{c_1}, \quad (4.123)$$

it follows that

$$\limsup_{t\to\infty} \mathbb{E}^{z_0}[\|z(t)\|^2] \leq \frac{c_0}{c_1 \lambda_{\min}(\tilde{P})}, \quad (4.124)$$

and, since $\bar{e}_x(t) = (\mathfrak{L}_1 \otimes I_n)\varepsilon(t)$, (4.124) implies

$$\limsup_{t\to\infty} \mathbb{E}^{\varepsilon_0}[\|\varepsilon(t)\|^2] \leq \frac{c_0}{c_1 \lambda_{\min}(\tilde{P}) \lambda_{\min}(\mathfrak{L}_1)}, \quad (4.125)$$

which further implies that the pathwise trajectory of the state tracking error for each agent of the closed-loop system associated with the plant dynamics is uniformly ultimately bounded in a mean-square sense. □

Remark 4.1. It is worth pointing out that the output feedback control algorithm presented in Theorem 4.3 is neither a direct nor a simple extension of the state feedback control algorithm given in Theorem 4.2. Namely, the output feedback algorithm requires the design of the observer dynamics (4.96) involving the reconstructed state neighborhood synchronization error as well as the output neighborhood synchronization error.

4.11. Multiple Aircraft Consensus Control

In this section, we apply the state and output feedback control architectures developed in the paper to a representative example involving multiple aircraft consensus control. Specifically, we consider a multiagent system representing the controlled lateral dynamics of four follower aircrafts and one leader aircraft with a communication graph topology shown in Figure 4.11.1. Node L represents the leader aircraft and nodes 1 through 4 represent the follower aircrafts.

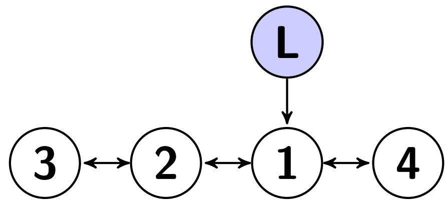


Figure 4.11.1: Leader-follower communication topology $\mathfrak{G}$. (See color figure online).

4.11.1. Output Feedback Control

Next, we consider an output feedback control architecture. For the leader aircraft, we take $x_{00} = [1, -2, 1]^{\mathrm{T}}$ and $g = [1,\ 1]^{\mathrm{T}}$, with the system matrices A and B as given by (4.79) and C given by

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{4.126}$$

Furthermore, we take the reference input

$$r_0(t) = \begin{bmatrix} 4.5551 & 0.1521 & -0.0810 \\ -7.6520 & -0.1964 & 0.1897 \end{bmatrix} x_0(t). \tag{4.127}$$

The follower aircraft dynamics are given by (4.88) and (4.89), where $x_1(t) \triangleq [\beta_1(t), p_1(t), r_1(t)]^{\mathrm{T}}$, $x_{10} = [4, -6, -2]^{\mathrm{T}}$, $x_2(t) \triangleq [\beta_2(t), p_2(t), r_2(t)]^{\mathrm{T}}$, $x_{20} = [6, 0, 2]^{\mathrm{T}}$, $x_3(t) \triangleq [\beta_3(t), p_3(t), r_3(t)]^{\mathrm{T}}$, $x_{30} = [4, -2, -2]^{\mathrm{T}}$, and $x_4(t) \triangleq [\beta_4(t), p_4(t), r_4(t)]^{\mathrm{T}}$, $x_{40} = [2, -2, 3]^{\mathrm{T}}$. The uncompromised control inputs for the follower aircrafts is once again $u_i(t) \triangleq \left[\delta_{\mathrm{ail},i}(t),\ \delta_{\mathrm{rud},i}(t)\right]^{\mathrm{T}}$, $i \in \{1, 2, 3, 4\}$. The actuator attacks are modeled by

$$\phi_1(t) = \begin{bmatrix} 0.1(1 - e^{-0.1t}) \\ 0.08(1 - e^{-0.15t}) \end{bmatrix}, \quad \phi_2(t) = \begin{bmatrix} 0.2(1 - e^{-0.1t}) \\ 0.1(1 - e^{-0.15t}) \end{bmatrix}, \tag{4.128}$$

$$\phi_3(t) = \begin{bmatrix} 0.15(1 - e^{-0.1t}) \\ 0.1(1 - e^{-0.15t}) \end{bmatrix}, \quad \phi_4(t) = \begin{bmatrix} 0.2(1 - e^{-0.1t}) \\ 0.08(1 - e^{-0.15t}) \end{bmatrix}, \tag{4.129}$$

and the output neighborhood synchronization error attacks are modeled as

$$d_{y1}(t) = \begin{bmatrix} \sin(2t) \\ \sin(2t) \end{bmatrix}, \quad d_{y2}(t) = \begin{bmatrix} 5\cos(2.5t) \\ 5\cos(2.5t) \end{bmatrix}, \quad d_{y3}(t) = \begin{bmatrix} 2\sin(3t) \\ 2\sin(3t) \end{bmatrix}, \quad d_{y4}(t) = \begin{bmatrix} \cos(2t) \\ \cos(2t) \end{bmatrix}. \tag{4.130}$$

To design a distributed output feedback adaptive controller we use Theorem 4.3 with

$$P = \begin{bmatrix} 24.1729 & 0.0675 & -0.0223 \\ 0.0675 & 0.0103 & 0.0008 \\ -0.0223 & 0.0008 & 0.0007 \end{bmatrix}, \quad Q = \begin{bmatrix} 4602.4 & -97.7 & 350.1 \\ -97.7 & 2.7 & -6.3 \\ 350.1 & -6.3 & 30.7 \end{bmatrix} \tag{4.131}$$

and design parameters $\tau = 1$, $\gamma = 10$, $c = 10$, $n_{d_{yi}} = 4$, $i \in \{1, 2, 3, 4\}$, $\sigma_{d_{yi}} = 3$, $i \in \{1, 2, 3, 4\}$, $n_{\phi_i} = 4$, $i \in \{1, 2, 3, 4\}$, and $\sigma_{\phi_i} = 2$, $i \in \{1, 2, 3, 4\}$.

First, we use the architecture proposed in [131, 132, 142, 143] for designing a consensus controller without any corrective action in the presence of sensor and actuator attacks. This design is shown in Figures 4.11.2-4.11.5 for the ith follower agent, where $i \in \{1, 2, 3, 4\}$, which show a sample trajectory along with the standard deviation of the state tracking error $\varepsilon_i(t) = x_i(t) - x_0(t)$ for agent $i \in \{1, 2, 3, 4\}$ versus time for 10 sample paths. The mean control profile is also plotted in Figures 4.11.2-4.11.5. It is clear from the Figures 4.11.2-4.11.5 that in the presence of sensor and actuator attacks a consensus control architecture predicted on a nominal neighborhood synchronization error is unable to guarantee consensus.

Alternatively, the system performance of the controller given by (4.95) with the proposed adaptive scheme is shown in Figures 4.11.6-4.11.9 for the ith follower agent, where $i \in \{1, 2, 3, 4\}$. Specifically, Figures 4.11.6-4.11.9 show a sample trajectory along with the standard deviation of the state tracking error $\varepsilon_i(t) = x_i(t) - x_0(t)$ for agent $i \in \{1, 2, 3, 4\}$ versus time for 10 sample paths. The mean control profile is also plotted in Figures 4.11.6-4.11.9. It follows from Theorem 4.3 that the state tracking error for each agent is guaranteed to be uniformly ultimate bounded in a mean-square sense.

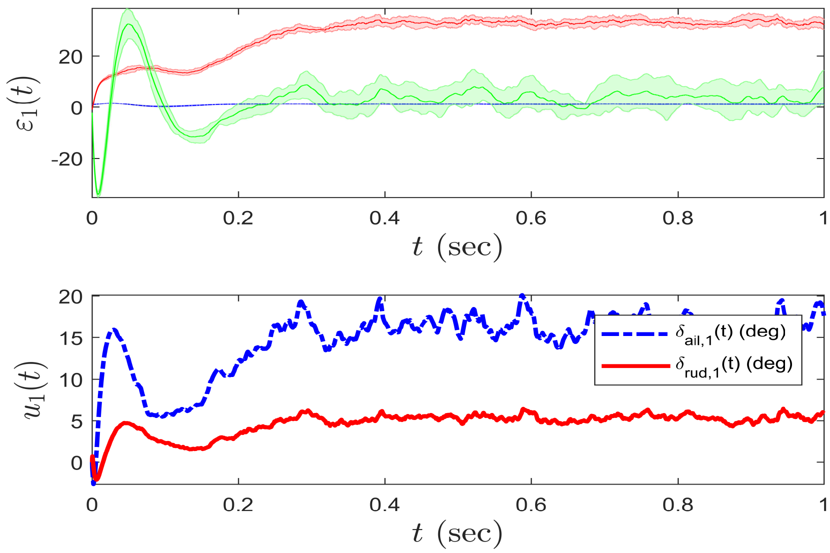


Figure 4.11.2: Agent 1 state tracking error sample trajectory without any corrective actions, with the sample standard deviation of the closed-loop nominal error trajectories versus time; $\beta_1(t) - \beta(t)$ in blue, $p_1(t) - p(t)$ in red, and $r_1(t) - r(t)$ in green. The control profile is plotted as the mean of the 10 sample runs.

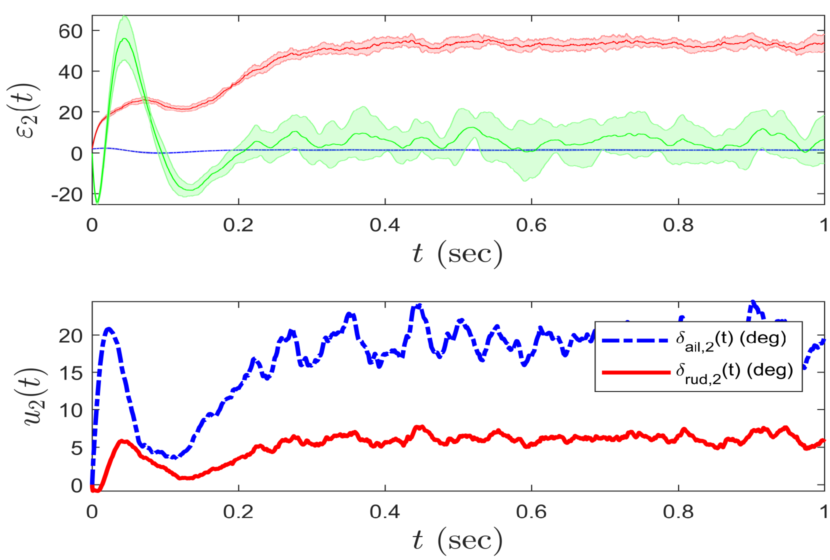


Figure 4.11.3: Agent 2 state tracking error sample trajectory without any corrective actions, with the sample standard deviation of the closed-loop nominal error trajectories versus time; $\beta_2(t) - \beta(t)$ in blue, $p_2(t) - p(t)$ in red, and $r_2(t) - r(t)$ in green. The control profile is plotted as the mean of the 10 sample runs.

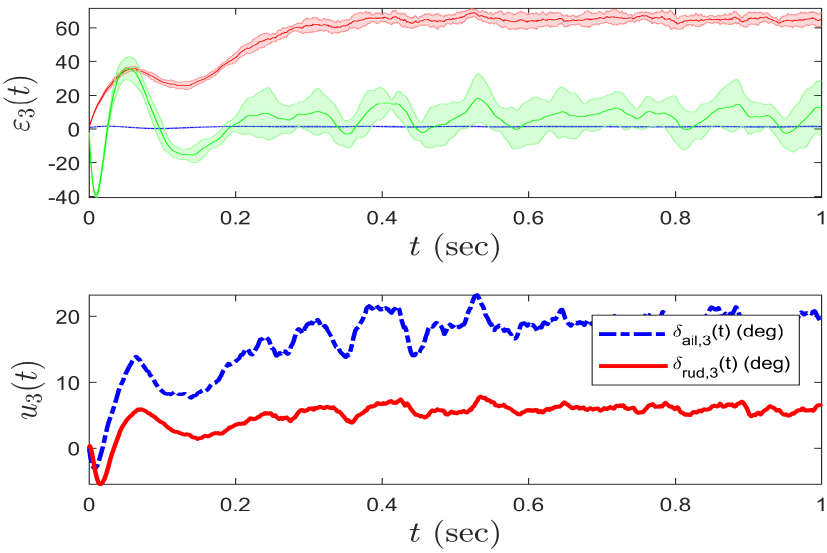


Figure 4.11.4: Agent 3 state tracking error sample trajectory without any corrective actions, with the sample standard deviation of the closed-loop nominal error trajectories versus time; $\beta_3(t) - \beta(t)$ in blue, $p_3(t) - p(t)$ in red, and $r_3(t) - r(t)$ in green. The control profile is plotted as the mean of the 10 sample runs.

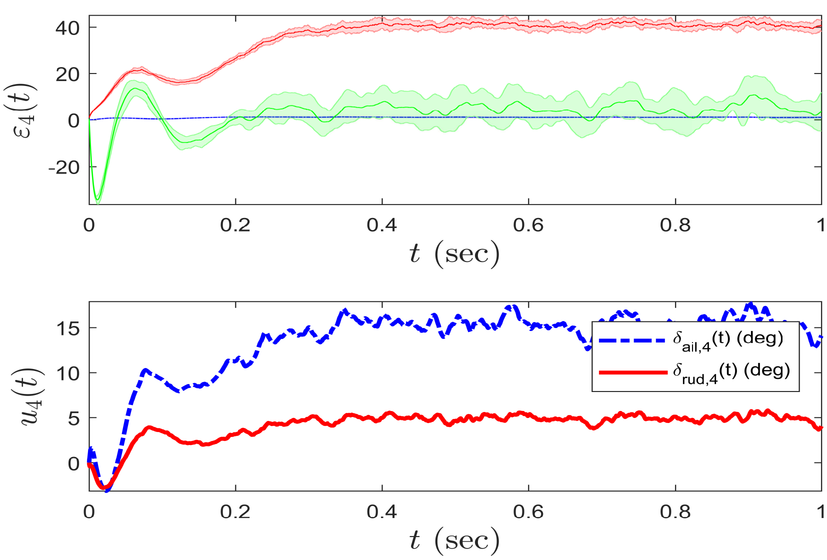


Figure 4.11.5: Agent 4 state tracking error sample trajectory without any corrective actions, with the sample standard deviation of the closed-loop nominal error trajectories versus time; $\beta_4(t) - \beta(t)$ in blue, $p_4(t) - p(t)$ in red, and $r_4(t) - r(t)$ in green. The control profile is plotted as the mean of the 10 sample runs.

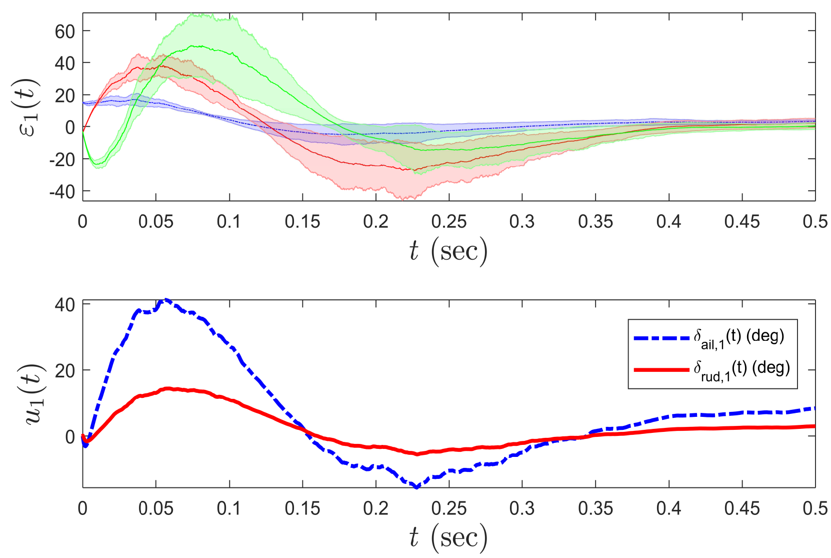


Figure 4.11.6: Agent 1 state tracking error sample trajectory with the sample standard deviation of the closed-loop nominal error trajectories versus time; $\beta_1(t) - \beta(t)$ in blue, $p_1(t) - p(t)$ in red, and $r_1(t) - r(t)$ in green. The control profile is plotted as the mean of the 10 sample runs.

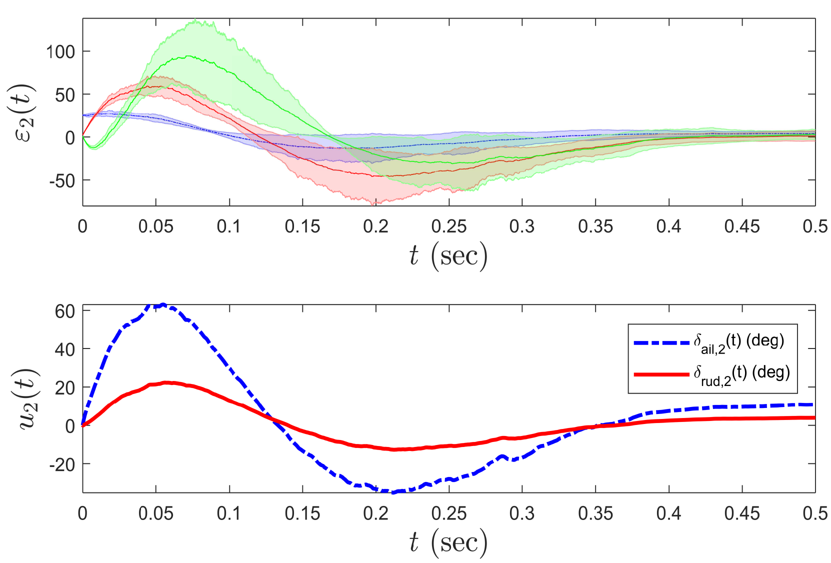


Figure 4.11.7: Agent 2 state tracking error sample trajectory with the sample standard deviation of the closed-loop nominal error trajectories versus time; $\beta_2(t) - \beta(t)$ in blue, $p_2(t) - p(t)$ in red, and $r_2(t) - r(t)$ in green. The control profile is plotted as the mean of the 10 sample runs.

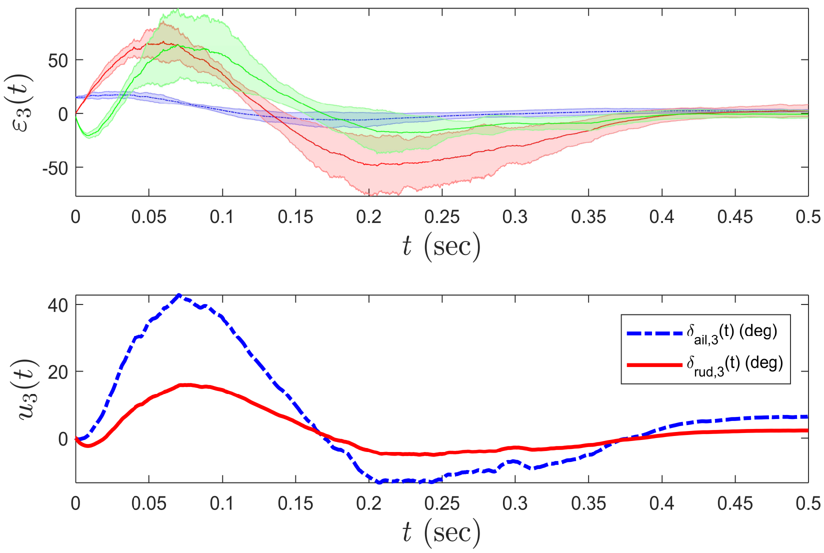


Figure 4.11.8: Agent 3 state tracking error sample trajectory with the sample standard deviation of the closed-loop nominal error trajectories versus time; $\beta_3(t) - \beta(t)$ in blue, $p_3(t) - p(t)$ in red, and $r_3(t) - r(t)$ in green. The control profile is plotted as the mean of the 10 sample runs.

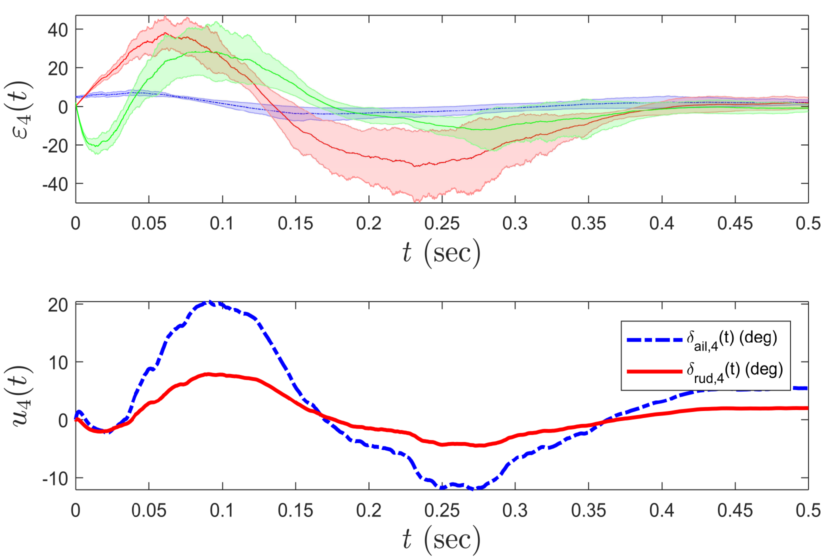


Figure 4.11.9: Agent 4 state tracking error sample trajectory with the sample standard deviation of the closed-loop nominal error trajectories versus time; $\beta_4(t) - \beta(t)$ in blue, $p_4(t) - p(t)$ in red, and $r_4(t) - r(t)$ in green. The control profile is plotted as the mean of the 10 sample runs.

Chapter 5

Energy-Based Feedback Control for Stochastic Port-Controlled Hamiltonian Systems

5.1. Introduction

In this chapter, we use the stochastic stability and dissipativity framework developed in [111], [114] to extend the deterministic passivity-based control framework for port-controlled Hamiltonian systems of [100–102] to nonlinear stochastic port-controlled Hamiltonian systems. Specifically, an energy-based control framework for stochastic port-controlled Hamiltonian systems is developed using a stochastic controller design methodology that achieves stabilization via stochastic system passivation. The interconnection and damping matrix functions of the stochastic port-controlled Hamiltonian system are shaped so that the physical (Hamiltonian) system structure is preserved at the closed-loop level and the closed-loop average energy function is equal to the difference between the average physical energy of the system and the average energy supplied by the controller. Since the Hamiltonian structure is preserved at the closed-loop level, the passivity-based stochastic controller is *robust* with respect to unmodeled passive dynamics. Passivity-based control architectures are extremely appealing since the control action has a clear *physical* energy interpretation, which can considerably simplify controller implementation.

In addition, we consider energy-based *dynamic* controllers for stochastic port-controlled Hamiltonian systems, wherein energy shaping is achieved by combining the physical energy of the plant and the emulated energy of the feedback controller. For deterministic systems, this approach has been extensively studied by Ortega *et al.* [97], [98] to design Euler-Lagrange controllers for potential energy shaping of mechanical systems. The efficacy of the proposed framework is highlighted on several illustrative numerical examples involving an inverted pendulum and a pair of undamped coupled oscillators.

5.2. Stochastic Port-Controlled Hamiltonian Systems

In this section, we consider the stochastic port-controlled Hamiltonian system given by

$$\mathrm{d}x(t) = \left([J(x(t)) - R(x(t))] \left(\frac{\partial H}{\partial x}(x(t)) \right)^{\mathrm{T}} + G(x(t))u(t) \right) \mathrm{d}t + D(x)\mathrm{d}w(t),$$

$$x(t_0) \overset{\text{a.s.}}{=} x_0, \ t \geq t_0, \quad (5.1)$$

$$y(t) = h(x(t)) = G^{\mathrm{T}}(x(t)) \left(\frac{\partial H}{\partial x}(x(t)) \right)^{\mathrm{T}}, \quad (5.2)$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$, $\mathcal{D}$ is an open set with $x_{\mathrm{e}} \in \mathcal{D}$, $u(t) \in \mathcal{H}_m^U$, $U \subseteq \mathbb{R}^m$, $y(t) \in \mathcal{H}_m^Y$, $Y \subseteq \mathbb{R}^m$, $H : \mathcal{D} \to \mathbb{R}$ is a two-times differentiable *Hamiltonian function* for the stochastic system (5.1) and (5.2), $J : \mathcal{D} \to \mathbb{R}^{n \times n}$ is such that $J(x) = -J^{\mathrm{T}}(x)$, $R : \mathcal{D} \to \mathbb{R}^{n \times n}$ is such that $R(x) \geq 0$, $x \in \mathcal{D}$, $[J(x) - R(x)] \left(\frac{\partial H}{\partial x}(x) \right)^{\mathrm{T}}$, $x \in \mathcal{D}$, is Lipschitz continuous, $G : \mathcal{D} \to \mathbb{R}^{n \times m}$ and $h : \mathcal{D} \to \mathbb{R}^m$ are continuous, $w(t)$ is a d-dimensional independent standard Wiener process, $x(t_0)$ is independent of $(w(t) - w(t_0)), t \geq t_0$, and $D : \mathcal{D} \to \mathbb{R}^{n \times d}$ is continuous and satisfies $D(x_{\mathrm{e}}) = 0$ for every equilibrium point $x_{\mathrm{e}} \in \mathcal{D}$ of (5.1). The skew-symmetric matrix function $J(x)$, $x \in \mathcal{D}$, captures the internal system interconnection structure, the input matrix function $G(x)$, $x \in \mathcal{D}$, captures interconnections with the environment, and the symmetric nonnegative definite matrix function $R(x)$, $x \in \mathcal{D}$, captures system dissipation.

Here we assume that $u(\cdot)$ satisfies sufficient regularity conditions such that (5.1) has a unique solution forward in time. Specifically, we assume that the control process $u(\cdot)$ in (5.1) is restricted to the class of *admissible* controls consisting of measurable functions

$u(\cdot)$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ such that $u(t) \in \mathcal{H}_m$, $t \geq t_0$, and, for all $t \geq s$, $w(t) - w(s)$ is independent of $u(\tau)$, $w(\tau)$, $\tau \leq s$, and $x(0)$, and hence, $u(\cdot)$ is nonanticipative. Furthermore, we assume $u(\cdot)$ takes values in a compact, metrizable set $\mathcal{U}$ and the uniform Lipschitz continuity and growth conditions (3.3) and (3.4) hold for the port-controlled drift and diffusion terms in (5.1) uniformly in u. In this case, it follows from Theorem 2.2.4 of [4] that there exists a pathwise unique solution to (5.1) in $(\Omega, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P}^{x_0})$.

A measurable function $\phi : \mathcal{D} \to U$ satisfying $\phi(x_\mathrm{e}) = 0$ is called a *control law*. If $u(t) = \phi(x(t))$, $t \geq t_0$, where $\phi(\cdot)$ is a control law and $x(t)$, $t \geq t_0$, satisfies (5.1), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot)$ has values in U.

Next, we provide constructive sufficient conditions for energy-based feedback control of stochastic port-controlled Hamiltonian systems. Specifically, we seek feedback controllers $u(t) = \phi(x(t))$, $t \geq t_0$, where $\phi : \mathcal{D} \to U$, such that the closed-loop system has the form

$$\begin{aligned}\mathrm{d}x(t) &= \left([J(x(t)) - R(x(t))]\left(\frac{\partial H}{\partial x}(x(t))\right)^{\mathrm{T}} + G(x(t))\phi(x(t))\right)\mathrm{d}t + D(x)\mathrm{d}w(t)\\ &= \left([J_\mathrm{s}(x(t)) - R_\mathrm{s}(x(t))]\left(\frac{\partial H_\mathrm{s}}{\partial x}(x(t))\right)^{\mathrm{T}}\right)\mathrm{d}t + D(x)\mathrm{d}w(t), \quad x(t_0) \overset{\text{a.s.}}{=} x_0, \quad t \geq t_0,\end{aligned} \tag{5.3}$$

where $H_\mathrm{s} : \mathcal{D} \to \mathbb{R}$ is a *shaped Hamiltonian function* for the closed-loop system (5.3), $J_\mathrm{s} : \mathcal{D} \to \mathbb{R}^{n\times n}$ is a shaped interconnection matrix function for the closed-loop system and satisfies $J_\mathrm{s}(x) = -J_\mathrm{s}^{\mathrm{T}}(x)$, and $R_\mathrm{s} : \mathcal{D} \to \mathbb{R}^{n\times n}$ is a shaped dissipation matrix function for the closed-loop system and satisfies $R_\mathrm{s}(x) \geq 0$, $x \in \mathcal{D}$.

Theorem 5.1. Consider the nonlinear stochastic port-controlled Hamiltonian system given by (5.1). Assume there exist functions $\phi : \mathcal{D} \to U$, H_s, $H_\mathrm{c} : \mathcal{D} \to \mathbb{R}$, J_s, $J_\mathrm{a} : \mathcal{D} \to \mathbb{R}^{n\times n}$, R_s, $R_\mathrm{a} : \mathcal{D} \to \mathbb{R}^{n\times n}$ such that $H_\mathrm{s}(x) = H(x) + H_\mathrm{c}(x)$ is two-times continuously differentiable, $J_\mathrm{s}(x) = J(x) + J_\mathrm{a}(x)$, $J_\mathrm{s}(x) = -J_\mathrm{s}^{\mathrm{T}}(x)$, $R_\mathrm{s}(x) = R(x) + R_\mathrm{a}(x)$, $R_\mathrm{s}(x) = R_\mathrm{s}^{\mathrm{T}}(x) \geq 0$, $x \in \mathcal{D}$, and

$$\frac{\partial H_{\rm c}}{\partial x}(x_{\rm e}) = -\frac{\partial H}{\partial x}(x_{\rm e}), \quad x_{\rm e} \in \mathcal{D}, \tag{5.4}$$

$$\frac{\partial^2 H_{\rm c}}{\partial x^2}(x_{\rm e}) > -\frac{\partial^2 \mathcal{H}}{\partial x^2}(x_{\rm e}), \quad x_{\rm e} \in \mathcal{D}, \tag{5.5}$$

$$[J_{\rm s}(x) - R_{\rm s}(x)]\left(\frac{\partial H_{\rm c}}{\partial x}(x)\right)^{\rm T} = -[J_{\rm a}(x) - R_{\rm a}(x)]\left(\frac{\partial H}{\partial x}(x)\right)^{\rm T} + G(x)\phi(x), \ x \in \mathcal{D}, \tag{5.6}$$

$$\frac{1}{2}{\rm tr}\left(D^{\rm T}(x)\frac{\partial^2 H_{\rm s}}{\partial x^2}(x)D(x)\right) \le \frac{\partial H_{\rm s}}{\partial x}(x)R_{\rm s}(x)\left(\frac{\partial H_{\rm s}}{\partial x}(x)\right)^{\rm T}, \quad x \in \mathcal{D}. \tag{5.7}$$

Then the equilibrium solution $x(t) \overset{\rm a.s.}{\equiv} x_{\rm e}$ of the closed-loop system (5.3) is Lyapunov stable in probability. If, in addition, $\{x \in \mathcal{D} : \frac{\partial H_{\rm s}}{\partial x}(x)R_{\rm s}(x)\left(\frac{\partial H_{\rm s}}{\partial x}(x)\right)^{\rm T} - \frac{1}{2}{\rm tr}\left(D^{\rm T}(x)\frac{\partial^2 H_{\rm s}}{\partial x^2}(x)D(x)\right) = 0\} = \{x_{\rm e}\}$, then the equilibrium solution $x(t) \overset{\rm a.s.}{\equiv} x_{\rm e}$ of the closed-loop system (5.3) is locally asymptotically stable in probability.

Proof. Condition (5.5) implies that with feedback controller $u(t) = \phi(x(t))$ the closed-loop system (5.1) has the Hamiltonian structure given by (5.3). Furthermore, it follows from (5.4) and (5.5) that the energy function $H_{\rm s}(\cdot)$ has a local minimum at $x = x_{\rm e}$. Hence, $x = x_{\rm e}$ is an equilibrium point of the closed-loop system.

Next, consider the Lyapunov function candidate for the closed-loop system (5.3) given by $V(x) = H_{\rm s}(x) - H_{\rm s}(x_{\rm e})$. Now, the corresponding infinitesimal generator $\mathcal{L}V(x)$ of $V(x)$ is given by

$$\begin{aligned}\mathcal{L}V(x) &= \mathcal{L}H_{\rm s}(x) \\ &= \frac{\partial H_{\rm s}}{\partial x}(x)\left([J(x) - R(x)]\left(\frac{\partial H}{\partial x}(x)\right)^{\rm T} + G(x)\phi(x)\right) + \frac{1}{2}{\rm tr}\left(D^{\rm T}(x)\frac{\partial^2 H_{\rm s}}{\partial x^2}(x)D(x)\right) \\ &= -\frac{\partial H_{\rm s}}{\partial x}(x)R_{\rm s}(x)\left(\frac{\partial H_{\rm s}}{\partial x}(x)\right)^{\rm T} + \frac{1}{2}{\rm tr}\left(D^{\rm T}(x)\frac{\partial^2 H_{\rm s}}{\partial x^2}(x)D(x)\right) \\ &\le 0, \quad x \in \mathcal{D}, \end{aligned} \tag{5.8}$$

and hence, it follows from Theorem 2.1 that the closed-loop system (5.3) is Lyapunov stable in probability.

Finally, it follows from Lyapunov stability in probability that $\mathcal{B}_\varepsilon(x_e) \in \mathcal{D}$, $\varepsilon > 0$, is positively invariant as $\varepsilon \to 0$, and hence, asymptotic stability in probability of the closed-loop system follows immediately from [87, Cor. 4.1] with

$$\begin{aligned}\eta(V(x)) &\triangleq \frac{\partial V}{\partial x}(x)R_s(x)\left(\frac{\partial V}{\partial x}(x)\right)^{\mathrm{T}} - \frac{1}{2}\mathrm{tr}\left(D^{\mathrm{T}}(x)\frac{\partial^2 V}{\partial x^2}(x)D(x)\right)\\ &= \frac{\partial H_s}{\partial x}(x)R_s(x)\left(\frac{\partial H_s}{\partial x}(x)\right)^{\mathrm{T}} - \frac{1}{2}\mathrm{tr}\left(D^{\mathrm{T}}(x)\frac{\partial^2 H_s}{\partial x^2}(x)D(x)\right).\end{aligned}$$

This completes the proof. □

Theorem 5.1 presents constructive sufficient conditions for feedback stabilization that preserve the physical Hamiltonian structure at the closed-loop level while providing a shaped Hamiltonian energy function as a Lyapunov function for the closed-loop system. These sufficient conditions consist of a partial differential equation parameterized by the auxiliary energy function $H_c(\cdot)$, the auxiliary interconnection matrix function $J_a(\cdot)$, and auxiliary dissipation matrix functions $R_a(\cdot)$, and their solution characterize the set of all desired shaped energy functions that can be assigned while preserving the system Hamiltonian structure at the closed-loop level.

To apply Theorem 5.1, we fix the structure of the interconnection $J_s(\cdot)$ and dissipation $R_s(\cdot)$ matrix functions and solve for the closed-loop energy function $H_s(\cdot)$ such that (5.7) holds. Although in this case solving (5.6) appears formidable, it is in fact quite tractable since the partial differential equation (5.6) is parameterized via the interconnection and dissipation matrix functions, which can be chosen by the control designer to satisfy system physical constraints.

More specifically, we can fix the interconnection and dissipation matrix functions $J_a(x)$ and $R_a(x)$ (e.g., initially they can be taken to be zero) and solve the partial differential equation

$$G^{\perp}(x)[J(x) + J_a(x) - (R(x) + R_a(x))]\frac{\partial H_a}{\partial x}(x) = -G^{\perp}(x)[J_a(x) - R_a(x)]\frac{\partial H}{\partial x}(x),$$

where $G^{\perp}(x)$ is a left annihilator of $G(x)$, in terms of $H_{\mathrm{a}}(x)$. This results in a *linear* partial differential equation of the form $A(x)\left(\frac{\partial H_{\mathrm{a}}}{\partial x}\right)(x) = q(x)$ for which powerful solution techniques exist (e.g., the method of characteristics). In this case, if rank $G(x) = m$ and $\operatorname{rank}[G(x)\ b(x)] = \operatorname{rank} G(x) = m$, where

$$b(x) = [J_{\mathrm{s}}(x) - R_{\mathrm{s}}(x)]\left(\frac{\partial H_{\mathrm{c}}}{\partial x}(x)\right)^{\mathrm{T}} + [J_{\mathrm{a}}(x) - R_{\mathrm{a}}(x)]\left(\frac{\partial H}{\partial x}(x)\right)^{\mathrm{T}}, \tag{5.9}$$

then an explicit expression for the stabilizing feedback controller satisfying (5.6) is given by $\phi(x) = (G^{\mathrm{T}}(x)G(x))^{-1}G^{\mathrm{T}}(x)b(x)$, $x \in \mathcal{D}$. Alternatively, if $\operatorname{rank}[G(x)\ b(x)] = \operatorname{rank} G(x) < m$, $x \in \mathcal{D}$, then the feedback controller $\phi(x) = G^{\dagger}(x)b(x) + [I_m - G^{\dagger}(x)G(x)]z$, $x \in \mathcal{D}$, where $(\cdot)^{\dagger}$ denotes the Moore-Penrose generalized inverse and $z \in \mathbb{R}^m$, satisfies (5.6).

If there does not exist physical considerations for choosing $J_{\mathrm{a}}(x)$ and $R_{\mathrm{a}}(x)$, then they can be chosen to simplify the solution to the partial differential equation. This procedure is similar to the procedure detailed in [99] in designing globally stabilizing interconnection and damping assignment for deterministic port-controlled Hamiltonian systems. Alternatively, we can fix the shaped Hamiltonian $H_{\mathrm{s}}(\cdot)$ and solve for the interconnection and dissipation matrix functions. In this case, we do not need to solve a partial differential equation but rather an algebraic equation.

Assuming that the Hamiltonian energy function $H(\cdot)$ is lower bounded, it can be shown that stochastic port-controlled Hamiltonian systems provide an energy balance in terms of the stored or accumulated average energy, supplied average system energy, and dissipated energy. Specifically, computing the infinitesimal generator $\mathcal{L}H(x)$ of the Hamiltonian $H(x)$ yields the energy conservation equation

$$\mathcal{L}H(x) = u^{\mathrm{T}}h(x) - \frac{\partial H}{\partial x}(x)R(x)\left(\frac{\partial H}{\partial x}(x)\right)^{\mathrm{T}} + \frac{1}{2}\mathrm{tr}\left(D^{\mathrm{T}}(x)\frac{\partial^2 H}{\partial x^2}(x)D(x)\right), \quad (x,u) \in \mathcal{D} \times U. \tag{5.10}$$

Under certain conditions on the system dissipation, the energy-based controller given by Theorem 5.1 provides an energy balance of the controlled system. To see this, let $R_{\mathrm{a}}(x) \equiv 0$,

$R(x)\left(\frac{\partial H_{\rm c}}{\partial x}(x)\right)^{\rm T} = 0$, $x \in \mathcal{D}$, and $\frac{\partial^2 H_{\rm c}}{\partial x^2}(x)D(x) = 0$, $x \in \mathcal{D}$. In this case, the closed-loop dynamics are given by

$$\mathrm{d}x(t) = [J_{\rm s}(x(t)) - R(x(t))]\left(\frac{\partial H_{\rm s}}{\partial x}(x(t))\right)^{\rm T} + D(x(t))\mathrm{d}w(t), \quad x(t_0) \overset{\rm a.s.}{=} x_0, \quad t \geq t_0, \tag{5.11}$$

and hence,

$$\begin{aligned}\mathcal{L}H_{\rm s}(x) &= -\frac{\partial H_{\rm s}}{\partial x}(x)R(x)\left(\frac{\partial H_{\rm s}}{\partial x}(x)\right)^{\rm T} + \frac{1}{2}\mathrm{tr}\left(D^{\rm T}(x)\frac{\partial^2 H_{\rm s}}{\partial x^2}(x)D(x)\right)\\ &= -\frac{\partial H}{\partial x}(x)R(x)\left(\frac{\partial H}{\partial x}(x)\right)^{\rm T} + \frac{1}{2}\mathrm{tr}\left(D^{\rm T}(x)\frac{\partial^2 H}{\partial x^2}(x)D(x)\right), \quad x \in \mathcal{D}. \end{aligned}\tag{5.12}$$

Now, using (5.10), (5.12) yields

$$\mathcal{L}H_{\rm s}(x) = \mathcal{L}H(x) - u^{\rm T}h(x), \quad (x,u) \in \mathcal{D} \times U. \tag{5.13}$$

Finally, it follows from Dynkin's formula [94, theorem 7.12] that, for $t > \hat{t}$,

$$\mathbb{E}\left[H_{\rm s}(x(t))|\mathcal{F}_{\hat{t}}\right] = \mathbb{E}\left[H(x(t))|\mathcal{F}_{\hat{t}}\right] - \mathbb{E}\left[\int_{\hat{t}}^{t} u^{\rm T}(s)y(s)\mathrm{d}s|\mathcal{F}_{\hat{t}}\right] + \kappa, \tag{5.14}$$

where the integral in (5.14) is an Itô integral and $\kappa \triangleq H_{\rm s}(x(\hat{t})) - H(x(\hat{t}))$, which shows that the closed-loop average energy function $\mathbb{E}\left[H_{\rm s}(x(t))|\mathcal{F}_{\hat{t}}\right]$ is equal to the difference between the physical average energy $\mathbb{E}\left[H(x(t))|\mathcal{F}_{\hat{t}}\right]$ of the system and the average energy supplied by the controller modulo the constant κ.

5.3. Stochastic Port-Controlled Hamiltonian Systems: Dynamic Control

In this section, we consider energy-based dynamic control for stochastic port-controlled Hamiltonian systems, wherein energy shaping is achieved by combining the physical energy of the plant and the emulated energy of the controller. We begin by considering the port-controlled Hamiltonian system $\mathcal{G}$ given by (5.1) and (5.2) with $m = l$. Furthermore, we consider the port-controlled Hamiltonian feedback control system $\mathcal{G}_{\rm c}$ given by

$$\mathrm{d}x_{\rm c}(t) = \left([J_{\rm c}(x_{\rm c}(t)) - R_{\rm c}(x_{\rm c}(t))]\left(\frac{\partial H_{\rm c}}{\partial x_{\rm c}}(x_{\rm c}(t))\right)^{\rm T} + G_{\rm c}(x_{\rm c}(t))u_{\rm c}(t)\right)\mathrm{d}t$$

$$+D_{\mathrm{c}}(x_{\mathrm{c}}(t))\mathrm{d}w(t), \quad x_{\mathrm{c}}(t_0) \overset{\text{a.s.}}{=} x_{\mathrm{c}0}, \quad t \geq t_0, \tag{5.15}$$

$$y_{\mathrm{c}}(t) = h_{\mathrm{c}}(x_{\mathrm{c}}(t)) = G_{\mathrm{c}}^{\mathrm{T}}(x_{\mathrm{c}}(t))\left(\frac{\partial H_{\mathrm{c}}}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}}(t))\right)^{\mathrm{T}}, \tag{5.16}$$

where, for every $t \geq t_0$, $x_{\mathrm{c}}(t) \in \mathcal{H}_{n_{\mathrm{c}}}$, $u_{\mathrm{c}}(t) \in \mathcal{H}_{m_{\mathrm{c}}}^{U_{\mathrm{c}}}$, $y_{\mathrm{c}}(t) \in \mathcal{H}_{l_{\mathrm{c}}}^{Y_{\mathrm{c}}}$, $m_{\mathrm{c}} = l_{\mathrm{c}}$, $H_{\mathrm{c}} : \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}$ is a continuously differentiable Hamiltonian function of the feedback control system $\mathcal{G}_{\mathrm{c}}$, $J_{\mathrm{c}} : \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{c}} \times n_{\mathrm{c}}}$ is such that $J_{\mathrm{c}}(x_{\mathrm{c}}) = -J_{\mathrm{c}}^{\mathrm{T}}(x_{\mathrm{c}})$, $R_{\mathrm{c}} : \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{c}} \times n_{\mathrm{c}}}$ is such that $R_{\mathrm{c}}(x_{\mathrm{c}}) \geq 0$, $x_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}}}$, $[J_{\mathrm{c}}(x_{\mathrm{c}}) - R_{\mathrm{c}}(x_{\mathrm{c}})]\left(\frac{\partial H_{\mathrm{c}}}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}})\right)^{\mathrm{T}}$, $x_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}}}$, is Lipschitz continuous on $\mathbb{R}^{n_{\mathrm{c}}}$, $G_{\mathrm{c}} : \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{c}} \times m_{\mathrm{c}}}$, $D_{\mathrm{c}} : \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{c}} \times d}$, $m_{\mathrm{c}} = l$, and $l_{\mathrm{c}} = m$. Here, we assume that $u_{\mathrm{c}}(\cdot)$ is restricted to the class of admissible inputs consisting of measurable functions such that $u_{\mathrm{c}}(t) \in \mathcal{H}_{m_{\mathrm{c}}}^{U_{\mathrm{c}}}$ for all $t \geq t_0$.

The closed-loop dynamics can be written in Hamiltonian form given by

$$\begin{aligned} \mathrm{d}\tilde{x}(t) = {} & \left(\begin{bmatrix} J(x(t)) & -G(x(t))G_{\mathrm{c}}^{\mathrm{T}}(x_{\mathrm{c}}(t)) \\ G_{\mathrm{c}}(x_{\mathrm{c}}(t))G^{\mathrm{T}}(x(t)) & J_{\mathrm{c}}(x_{\mathrm{c}}(t)) \end{bmatrix} \right. \\ & \left. - \begin{bmatrix} R(x(t)) & 0 \\ 0 & R_{\mathrm{c}}(x_{\mathrm{c}}(t)) \end{bmatrix} \right) \begin{bmatrix} \left(\frac{\partial H}{\partial x}(x(t))\right)^{\mathrm{T}} \\ \left(\frac{\partial H_{\mathrm{c}}}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}}(t))\right)^{\mathrm{T}} \end{bmatrix} \\ & + \begin{bmatrix} D(x(t)) \\ D_{\mathrm{c}}(x_{\mathrm{c}}(t)) \end{bmatrix} \mathrm{d}w(t), \quad \tilde{x}(t_0) \overset{\text{a.s.}}{=} \tilde{x}_0, \quad t \geq t_0, \end{aligned} \tag{5.17}$$

where $\tilde{x} \triangleq [x^{\mathrm{T}}, x_{\mathrm{c}}^{\mathrm{T}}]^{\mathrm{T}}$. It can be seen from (5.17) that by relating the controller state variables x_{c} to the plant state variables x, one can shape the Hamiltonian function $H(\cdot) + H_{\mathrm{c}}(\cdot)$ so as to preserve the Hamiltonian structure under dynamic feedback for part of the closed-loop system associated with the plant dynamics. Since the closed-loop dynamical system (5.17) is Hamiltonian involving skew-symmetric interconnection matrix function terms and nonnegative-definite dissipation matrix function terms, we can establish the existence of energy-Casimir functions [16, 124] (i.e., dynamical invariants) that are independent of the closed-loop Hamiltonian and relate the controller states to the plant states. Furthermore, since the controller Hamiltonian $H_{\mathrm{c}}(\cdot)$ can be assigned, the energy-Casimir method can be used to construct suitable Lyapunov functions for the closed-loop system.

To proceed, consider the candidate vector energy-Casimir function $E : \mathcal{D} \times \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{c}}}$, where $E(\cdot,\cdot)$ is two-times continuously differentiable and has the form

$$E(x,x_{\mathrm{c}}) = F_1(x_{\mathrm{c}}) - F_2(x), \quad (x,x_{\mathrm{c}}) \in \mathcal{D} \times \mathbb{R}^{n_{\mathrm{c}}}, \tag{5.18}$$

where $F_1 : \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{c}}}$ and $F_2 : \mathcal{D} \to \mathbb{R}^{n_{\mathrm{c}}}$ are two-times continuously differentiable functions. Furthermore, we assume F_1 is a diffeomorphism on $\mathbb{R}^{n_{\mathrm{c}}}$ onto $\mathbb{R}^{n_{\mathrm{c}}}$, and hence, F_1^{-1} exists. Moreover, we assume that $\det \frac{\partial F_1}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}}) \neq 0$, $x_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}}}$. To ensure that the candidate vector energy-Casimir function $E(\cdot,\cdot)$ is constant along the pathwise trajectories of (5.17) we require that, for all $(x,x_{\mathrm{c}}) \in \mathcal{D} \times \mathbb{R}^{n_{\mathrm{c}}}$,

$$\mathrm{d}E(x,x_{\mathrm{c}}) = \mathrm{d}F_1(x_{\mathrm{c}}) - \mathrm{d}F_2(x) = 0. \tag{5.19}$$

Now, we can arrive at a set of *sufficient* conditions which guarantee that (5.19) holds. Specifically, it follows from (5.17) that (5.19) can be rewritten as

$$\begin{aligned}
\mathrm{d}E(x,x_{\mathrm{c}}) = {} & \begin{bmatrix} \Big[\frac{\partial F_1}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}})G_{\mathrm{c}}(x_{\mathrm{c}})G^{\mathrm{T}}(x) \\ -\frac{\partial F_2}{\partial x}(x)(J(x)-R(x))\Big]^{\mathrm{T}} \\ \Big[\frac{\partial F_1}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}})(J_{\mathrm{c}}(x_{\mathrm{c}})-R_{\mathrm{c}}(x_{\mathrm{c}})) \\ +\frac{\partial F_2}{\partial x}(x)G(x(t))G_{\mathrm{c}}^{\mathrm{T}}(x_{\mathrm{c}})\Big]^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \left(\frac{\partial H}{\partial x}(x)\right)^{\mathrm{T}} \\ \left(\frac{\partial H_{\mathrm{c}}}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}})\right)^{\mathrm{T}} \end{bmatrix} \mathrm{d}t \\
& + \left(\frac{1}{2}\mathrm{tr}\left(D_{\mathrm{c}}^{\mathrm{T}}(x_{\mathrm{c}})\frac{\partial^2 F_1}{\partial x_{\mathrm{c}}^2}(x_{\mathrm{c}})D_{\mathrm{c}}(x_{\mathrm{c}})\right) - \frac{1}{2}\mathrm{tr}\left(D^{\mathrm{T}}(x)\frac{\partial^2 F_2}{\partial x^2}(x)D(x)\right)\right)\mathrm{d}t, \\
& + \left(\frac{\partial F_1}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}})D_{\mathrm{c}}(x_{\mathrm{c}}) - \frac{\partial F_2}{\partial x}(x)D(x)\right)\mathrm{d}w, \quad (x,x_{\mathrm{c}}) \in \mathcal{D} \times \mathbb{R}^{n_{\mathrm{c}}}.
\end{aligned} \tag{5.20}$$

Hence, a set of sufficient conditions such that (5.19) holds is given by, for all $(x,x_{\mathrm{c}}) \in \mathcal{D}\times\mathbb{R}^{n_{\mathrm{c}}}$,

$$\frac{\partial F_1}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}})G_{\mathrm{c}}(x_{\mathrm{c}})G^{\mathrm{T}}(x) - \frac{\partial F_2}{\partial x}(x)(J(x)-R(x)) = 0, \tag{5.21}$$

$$\frac{\partial F_1}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}})(J_{\mathrm{c}}(x_{\mathrm{c}})-R_{\mathrm{c}}(x_{\mathrm{c}})) + \frac{\partial F_2}{\partial x}(x)G(x)G_{\mathrm{c}}^{\mathrm{T}}(x_{\mathrm{c}}) = 0, \tag{5.22}$$

$$\frac{\partial F_1}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}})D_{\mathrm{c}}(x_{\mathrm{c}}) - \frac{\partial F_2}{\partial x}(x)D(x) = 0, \tag{5.23}$$

$$\mathrm{tr}\left(D_{\mathrm{c}}^{\mathrm{T}}(x_{\mathrm{c}})\frac{\partial^2 F_1}{\partial x_{\mathrm{c}}^2}(x_{\mathrm{c}})D_{\mathrm{c}}(x_{\mathrm{c}})\right) - \mathrm{tr}\left(D^{\mathrm{T}}(x)\frac{\partial^2 F_2}{\partial x^2}(x)D(x)\right) = 0. \tag{5.24}$$

The following proposition summarizes the above results.

Proposition 5.1. Consider the feedback interconnection of the stochastic port-controlled Hamiltonian systems $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ given by (5.1) and (5.2), and (5.15) and (5.16), respectively. If there exist two-times continuously differentiable functions $F_1 : \mathbb{R}^{n_{\rm c}} \to \mathbb{R}^{n_{\rm c}}$ and $F_2 : \mathcal{D} \to \mathbb{R}^{n_{\rm c}}$, where $F_1(\cdot)$ is a diffeomorphism on $\mathbb{R}^{n_{\rm c}}$ onto $\mathbb{R}^{n_{\rm c}}$, $\det \frac{\partial F_1}{\partial x_{\rm c}}(x_{\rm c}) \neq 0$, $x_{\rm c} \in \mathbb{R}^{n_{\rm c}}$, such that, for all $(x, x_{\rm c}) \in \mathcal{D} \times \mathbb{R}^{n_{\rm c}}$,

$$\frac{\partial F_2}{\partial x}(x)J(x)\left(\frac{\partial F_2}{\partial x}(x)\right)^{\rm T} - \frac{\partial F_1}{\partial x_{\rm c}}(x_{\rm c})J_{\rm c}(x_{\rm c})\left(\frac{\partial F_1}{\partial x_{\rm c}}(x_{\rm c})\right)^{\rm T} = 0, \tag{5.25}$$

$$R_{\rm c}(x_{\rm c})\left(\frac{\partial F_1}{\partial x_{\rm c}}(x_{\rm c})\right)^{\rm T} = 0, \tag{5.26}$$

$$R(x)\left(\frac{\partial F_2}{\partial x}(x)\right)^{\rm T} = 0, \tag{5.27}$$

$$\frac{\partial F_2}{\partial x}(x)J(x) - \frac{\partial F_1}{\partial x_{\rm c}}(x_{\rm c})G_{\rm c}(x_{\rm c})G^{\rm T}(x) = 0, \tag{5.28}$$

$$\frac{\partial F_1}{\partial x_{\rm c}}(x_{\rm c})D_{\rm c}(x_{\rm c}) - \frac{\partial F_2}{\partial x}(x)D(x) = 0, \tag{5.29}$$

$$D_{\rm c}^{\rm T}(x_{\rm c})\frac{\partial^2 F_1}{\partial x_{\rm c}^2}(x_{\rm c})D_{\rm c}(x_{\rm c}) - D^{\rm T}(x)\frac{\partial^2 F_2}{\partial x^2}(x)D(x) = 0, \tag{5.30}$$

then

$$E(\tilde{x}(t)) = F_1(x_{\rm c}(t)) - F_2(x(t)) \overset{\rm a.s.}{=} c, \quad t \geq t_0, \tag{5.31}$$

where $c \in \mathbb{R}^{n_{\rm c}}$ and $\tilde{x}(t) = [x^{\rm T}(t),\, x_{\rm c}^{\rm T}(t)]^{\rm T}$ satisfies (5.17).

Note that conditions (5.25)–(5.30) provide sufficient conditions for guaranteeing that the vector energy-Casimir function $E(\cdot,\cdot)$ is constant along the pathwise trajectories of the closed-loop system (5.17). The constant vector $c \in \mathbb{R}^{n_{\rm c}}$ in (5.31) depends on the initial conditions for the plant and controller states.

If conditions (5.25)–(5.30) are satisfied, then the controller state variables along the trajectories of the closed-loop system given by (5.17) can be represented in terms of the plant state variables as $x_{\rm c} = F_1^{-1}(F_2(x) + c)$, $x \in \mathcal{D}$, $c \in \mathbb{R}^{n_{\rm c}}$. In this case, it follows that the closed-loop system associated with the plant dynamics is given by

$$\mathrm{d}x(t) = \left([J(x(t)) - R(x(t))]\left(\frac{\partial H}{\partial x}(x(t))\right)^{\rm T} - G(x(t))G_{\rm c}^{\rm T}(x_{\rm c}(t))\left(\frac{\partial H_{\rm c}}{\partial x_{\rm c}}(x_{\rm c}(t))\right)^{\rm T}\right)\mathrm{d}t$$

$$
\begin{aligned}
&\quad + D(x(t))\mathrm{d}w(t) \\
&= \left([J(x(t)) - R(x(t))] \left(\frac{\partial H}{\partial x}(x(t)) + \frac{\partial H_{\mathrm{c}}}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}}(t)) \frac{\partial x_{\mathrm{c}}}{\partial F_1(x_{\mathrm{c}})} \frac{\partial F_1}{\partial x}(x_{\mathrm{c}}(t)) \right)^{\mathrm{T}} \right) \mathrm{d}t \\
&\quad + D(x(t))\mathrm{d}w(t) \\
&= \left([J(x(t)) - R(x(t))] \left(\frac{\partial H_{\mathrm{s}}}{\partial x}(x(t)) \right)^{\mathrm{T}} \right) \mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(t_0) \overset{\text{a.s.}}{=} x_0, \quad t \geq t_0,
\end{aligned} \tag{5.32}
$$

where $H_{\mathrm{s}}(x) = H(x) + H_{\mathrm{c}}(F_1^{-1}(F_2(x) + c))$, $x \in \mathcal{D}$, is the *shaped* Hamiltonian function for the closed-loop system (5.32).

Next, we use the existence of the vector energy-Casimir function to construct stabilizing dynamic controllers that guarantee that the closed-loop system associated with the plant dynamics preserves the Hamiltonian structure without the need for solving a set of partial differential equations.

Theorem 5.2. Consider the feedback interconnection of the port-controlled Hamiltonian systems $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ given by (5.1) and (5.2), and (5.15) and (5.16), respectively. Assume that there exist two-times continuously differentiable functions $F_1 : \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{c}}}$ and $F_2 : \mathcal{D} \to \mathbb{R}^{n_{\mathrm{c}}}$, where $F_1(\cdot)$ is a diffeomorphism on $\mathbb{R}^{n_{\mathrm{c}}}$ onto $\mathbb{R}^{n_{\mathrm{c}}}$ and $\det \frac{\partial F_1}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}}) \neq 0$, $x_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}}}$, such that conditions (5.25)–(5.30) hold for all $(x, x_{\mathrm{c}}) \in \mathcal{D} \times \mathbb{R}^{n_{\mathrm{c}}}$, and assume that the Hamiltonian function $H_{\mathrm{c}} : \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}$ of the feedback controller $\mathcal{G}_{\mathrm{c}}$ is such that $H_{\mathrm{s}} : \mathcal{D} \to \mathbb{R}$ is given by $H_{\mathrm{s}}(x) = H(x) + H_{\mathrm{c}}(F_1^{-1}(F_2(x) + c))$, $x \in \mathcal{D}$. If

$$
\frac{\partial H_{\mathrm{c}}}{\partial x}(F_1^{-1}(F_2(x_{\mathrm{e}}) + c)) = -\frac{\partial H}{\partial x}(x_{\mathrm{e}}), \quad x_{\mathrm{e}} \in \mathcal{D}, \tag{5.33}
$$

$$
\frac{\partial^2 H_{\mathrm{c}}}{\partial x^2}(F_1^{-1}(F_2(x_{\mathrm{e}}) + c)) > -\frac{\partial^2 H}{\partial x^2}(x_{\mathrm{e}}), \quad x_{\mathrm{e}} \in \mathcal{D}, \tag{5.34}
$$

$$
\frac{1}{2}\mathrm{tr}\left(D^{\mathrm{T}}(x) \frac{\partial^2 H_{\mathrm{s}}}{\partial x^2}(x) D(x) \right) \leq \frac{\partial H_{\mathrm{s}}}{\partial x}(x) R(x) \left(\frac{\partial H_{\mathrm{s}}}{\partial x}(x) \right)^{\mathrm{T}}, \quad x \in \mathcal{D}, \tag{5.35}
$$

then the equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\mathrm{e}}$ of the closed-loop system associated with the plant dynamics (5.32) is Lyapunov stable in probability. If, in addition,

$$\left\{x \in \mathcal{D} : \frac{\partial H_{\mathrm{s}}}{\partial x}(x) R(x)\left(\frac{\partial H_{\mathrm{s}}}{\partial x}(x)\right)^{\mathrm{T}} - \frac{1}{2}\mathrm{tr}\left(D^{\mathrm{T}}(x)\frac{\partial^2 H_{\mathrm{s}}}{\partial x^2}(x) D(x)\right) = 0\right\} = \{x_{\mathrm{e}}\},$$

then the equilibrium solution $x(t) \stackrel{\text{a.s.}}{\equiv} x_{\mathrm{e}}$ of the closed-loop system associated with the plant dynamics (5.32) is locally asymptotically stable in probability.

Proof. The proof is similar to the proof of Theorem 5.1, and, hence, is omitted. □

5.4. Illustrative Numerical Examples

In this section, we provide several numerical examples to highlight the proposed stochastic energy-based feedback control framework.

Example 5.1. Consider the inverted pendulum with a stochastic state disturbance shown in Figure 5.4.1, where $m = 1\,\mathrm{kg}$ and $L = 1\,\mathrm{m}$. The system is governed by the dynamic equation of motion

$$\begin{bmatrix} \mathrm{d}x_1(t) \\ \mathrm{d}x_2(t) \end{bmatrix} = \left(\begin{bmatrix} x_2(t) \\ g\sin(x_1(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)\right)\mathrm{d}t + \begin{bmatrix} 0 \\ \sigma x_2(t) \end{bmatrix}\mathrm{d}w(t),$$
$$[x_1(t_0),\ x_2(t_0)]^{\mathrm{T}} \stackrel{\text{a.s.}}{=} [x_{10},\ x_{20}]^{\mathrm{T}}, \quad t \geq t_0, \qquad (5.36)$$

where $\sigma = 1$ is the variance of the stochastic disturbance $w(\cdot)$, g denotes the gravitational acceleration, $u(\cdot)$ is a control torque, $x_1 = \theta$, and $x_2 = \dot{\theta}$. Note that (5.36) can be written in the form of (5.1) with

$$J(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad R(x) = 0, \quad G(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D(x) = \begin{bmatrix} 0 \\ \sigma x_2 \end{bmatrix}, \qquad (5.37)$$

$\mathcal{D} = \mathbb{R}^2$, and Hamiltonian function $H(\cdot)$ corresponding to the total energy in the system given by $H(x) = \frac{x_2^2}{2} + g\cos x_1$.

Next, to stabilize the equilibrium point $x_{\mathrm{e}} = [\theta_{\mathrm{e}}, 0]^{\mathrm{T}}$ we assign the shaped Hamiltonian $H_{\mathrm{s}}(x) = \frac{x_2^2}{2} + \frac{1}{2}(x_1 - \theta_{\mathrm{e}})^2$ function for the closed-loop system. Furthermore, we set

$$J_{\mathrm{a}}(x) = 0, \quad R_{\mathrm{a}}(x) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad x \in \mathcal{D}. \qquad (5.38)$$

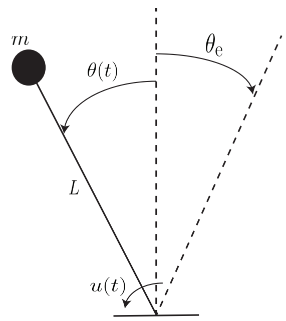


Figure 5.4.1: Inverted pendulum.

In this case, it follows from (5.6) that the feedback controller is given by $u = \phi(x) = -x_2 - (x_1 - \theta_{\rm e}) - g \sin x_1$, $x \in \mathcal{D}$. Next, note that $\mathcal{L}H_{\rm s}(x) = -0.5x_2^2 \leq 0$, $x \in \mathcal{D}$. Hence, $\{x \in \mathcal{D} : \eta(V(x)) = 0\} = \{x \in \mathcal{D} : x_2 = 0\}$. Finally, since, for every $x \in \mathcal{D}$, ${\rm d}x_2 \neq 0$ if and only $x_1 \neq \theta_{\rm e}$, it follows that $\{x \in \mathcal{D} : \eta(V(x)) = 0\} = \{x_{\rm e}\}$, and hence, the equilibrium solution $x(t) \equiv [\theta_{\rm e}, 0]^{\rm T}$ is asymptotically stable in probability. With $\theta_{\rm e} = 15^o$, Figure 5.4.2 shows the sample average along with the standard deviation of the angular displacement, angular velocity, and control profile versus time for $x(0) \overset{\rm a.s.}{=} [0.1745,\ -5]^{\rm T}$ for 10 sample paths. $\triangle$

Example 5.2. Consider the two-mass, two-spring system shown in Figure 5.4.3. A control force $\hat{u}(\cdot)$ acts on mass 2 with the goal to stabilize the position of the second mass. The system dynamics, with state variables defined in Figure 5.4.3, are given by

$$\begin{bmatrix} {\rm d}x_1(t) \\ {\rm d}x_2(t) \\ {\rm d}x_3(t) \\ {\rm d}x_4(t) \end{bmatrix} = \left(\begin{bmatrix} x_2(t) \\ -\frac{k_1+k_2}{m_1}x_1(t) + \frac{k_2}{m_1}x_3(t) \\ x_4(t) \\ -\frac{k_2}{m_2}(x_3(t) - x_1(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \frac{\hat{u}(t)}{m_2} \right) {\rm d}t + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sigma x_4(t) \end{bmatrix} {\rm d}w(t),$$

$$[x_1(t_0),\ x_2(t_0),\ x_3(t_0),\ x_4(t_0)]^{\rm T} \overset{\rm a.s.}{=} [x_{10},\ x_{20},\ x_{30},\ x_{40}]^{\rm T},\ t \geq t_0, \quad (5.39)$$

where $\sigma = 0.5$, $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$, and $x_4 = \dot{q}_2$. Note that (5.39) can be written in the form of (5.1) with $R(x) = 0$, $G(x) = [0,\ 0,\ 0,\ 1]^{\rm T}$, $u = \frac{\hat{u}}{m_2}$,

$$J(x) = \begin{bmatrix} 0 & \frac{1}{m_1} & 0 & 0 \\ -\frac{1}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{m_2} \\ 0 & 0 & -\frac{1}{m_2} & 0 \end{bmatrix}, \quad x \in \mathcal{D}, \quad (5.40)$$

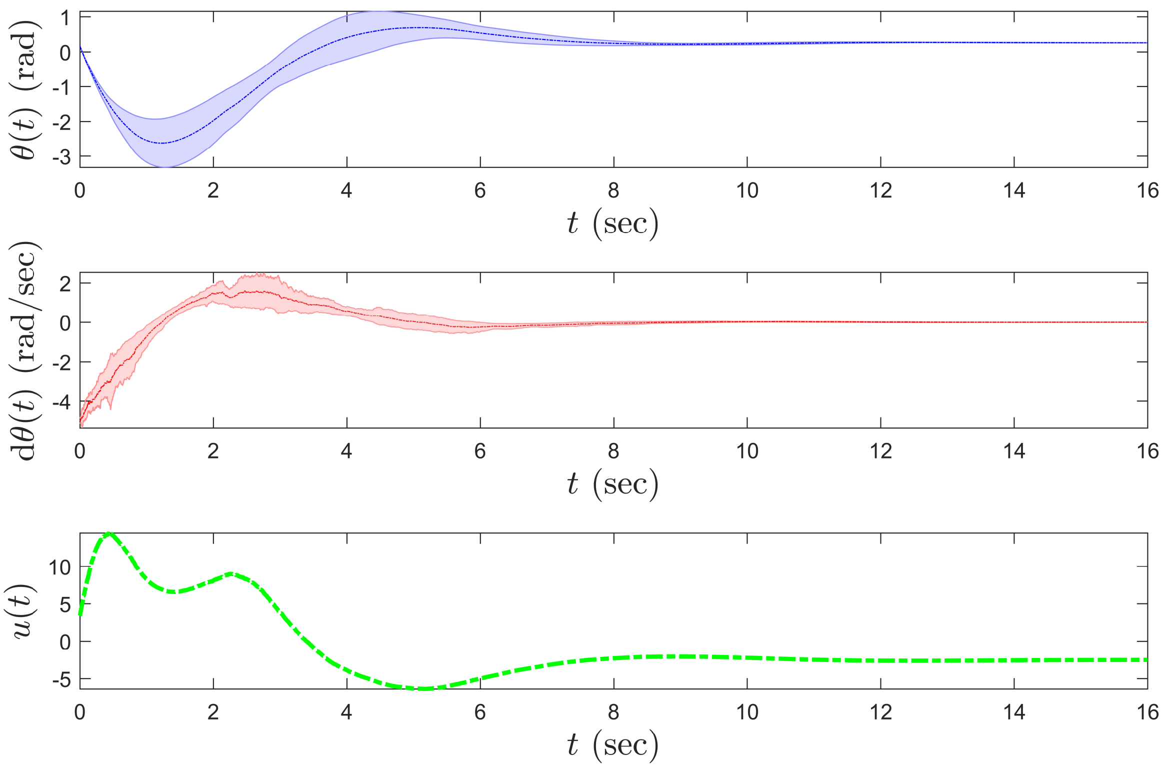


Figure 5.4.2: Angular displacement, angular velocity, and control profile versus time.

$\mathcal{D} = \{x \in \mathbb{R}^4 : x_1 \geq 0,\ x_3 \geq 0\}$, and Hamiltonian function $H(\cdot)$ corresponding to the total energy in the system given by $H(x) = \frac{m_1 x_2^2}{2} + \frac{m_2 x_4^2}{2} + \frac{k_1 x_1^2}{2} + \frac{k_2(x_3 - x_1)^2}{2}$.

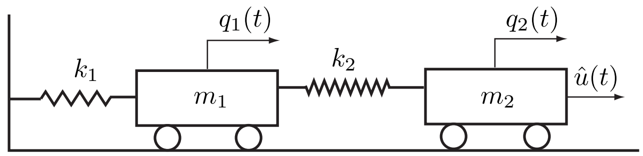


Figure 5.4.3: Two-mass, two-spring system.

Next, to stabilize the equilibrium point $x_{\rm e} = [x_{1\rm e},\ 0,\ x_{3\rm e},\ 0]^{\rm T}$, where $x_{1\rm e} = \frac{k_2}{(k_1+k_2)} x_{3\rm e}$, with steady-state control value of $u_{\rm c\,ss} = \frac{k_1 k_2}{m_2(k_1+k_2)} x_{3\rm e}$, we assign the shaped Hamiltonian function $H_{\rm s}(x) = \frac{m_1 x_2^2}{2} + \frac{m_2 x_4^2}{2} + \frac{k_1 x_1^2}{2} + \frac{k_2(x_3-x_1)^2}{2} - \frac{k_1 k_2}{(k_1+k_2)} x_{3\rm e} x_3$ for the closed-loop system. Furthermore, we set $J_{\rm a}(x) \equiv 0$ and

$$R_{\rm a}(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{m_2} \end{bmatrix}, \quad x \in \mathcal{D}. \tag{5.41}$$

In this case, it follows from (5.6) that the feedback controller is given by $u = \phi(x) = \frac{k_1 k_2}{m_2(k_1+k_2)} x_{3\rm e} - x_4$, $x \in \mathcal{D}$. Next, note that $\mathcal{L}H_{\rm s}(x) = -0.675 x_4^2 \leq 0$, $x \in \mathcal{D}$. Hence, $\{x \in \mathcal{D} : \eta(V(x)) = 0\} = \{x \in \mathcal{D} : x_4 = 0\}$, which implies that $x_1(t) - x_3(t) + \frac{k_1}{k_1+k_2} x_{3\rm e} = 0$ and $\dot{x}_3(t) = 0$, $t \geq 0$. In this case, it follows that $\dot{x}_1(t) = 0,\ t \geq 0$, and hence, $\dot{x}_2(t) = 0$, $t \geq 0$. Hence, the only point that belongs to $\{x \in \mathcal{D} : \eta(V(x)) = 0\} = \{x \in \mathcal{D} : x_4 = 0\}$ is $x_{\rm e} = [\frac{k_2}{(k_1+k_2)} x_{3\rm e},\ 0,\ x_{3\rm e},\ 0]^{\rm T}$, which implies that $x_{\rm e}$ is an asymptotically stable equilibrium point of the closed-loop system.

With $m_1 = 1.5\,\text{kg}$, $m_2 = 0.8\,\text{kg}$, $k_1 = 0.1\,\text{N/m}$, $k_2 = 0.3\,\text{N/m}$, $L = 0.4\,\text{m}$, and $x_{3\rm e} = 3\,\text{m}$, Figures 5.4.4 and 5.4.5 show, respectively, the sample average along with the standard deviation of the positions and velocities of the masses versus time for $[x_1(0),\ x_2(0),\ x_3(0),\ x_4(0)]^{\rm T} \stackrel{\rm a.s.}{=} [1,\ -0.5,\ 2.3,\ 0.5]^{\rm T}$ for 10 sample paths. Finally, Figure 5.4.6 shows the average control force versus time. $\triangle$

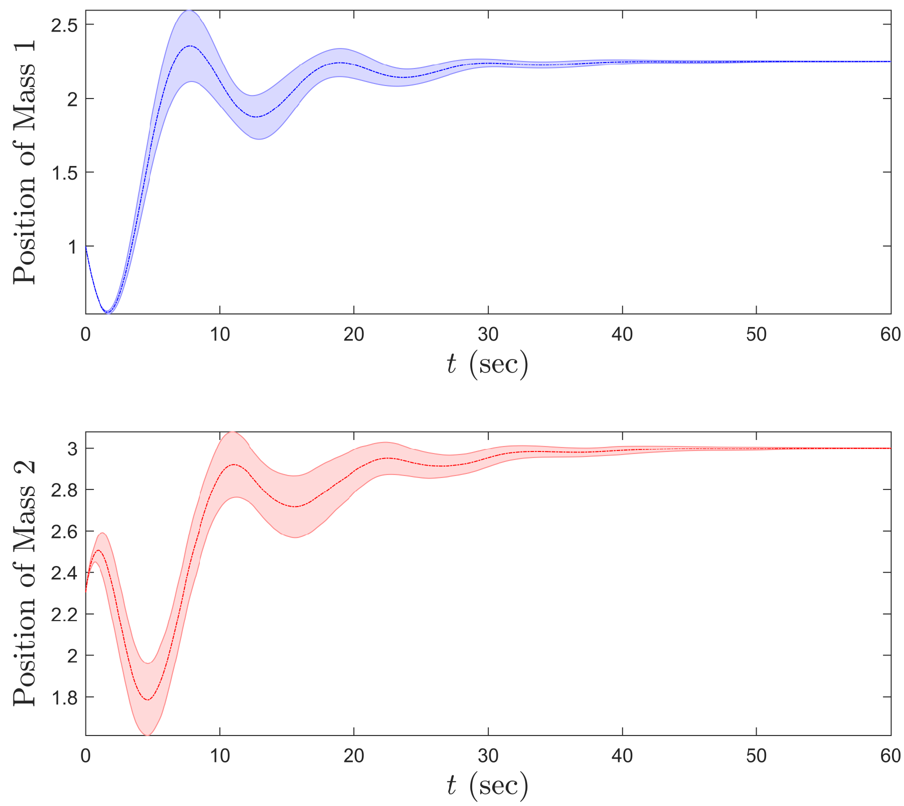


Figure 5.4.4: Mass positions versus time.

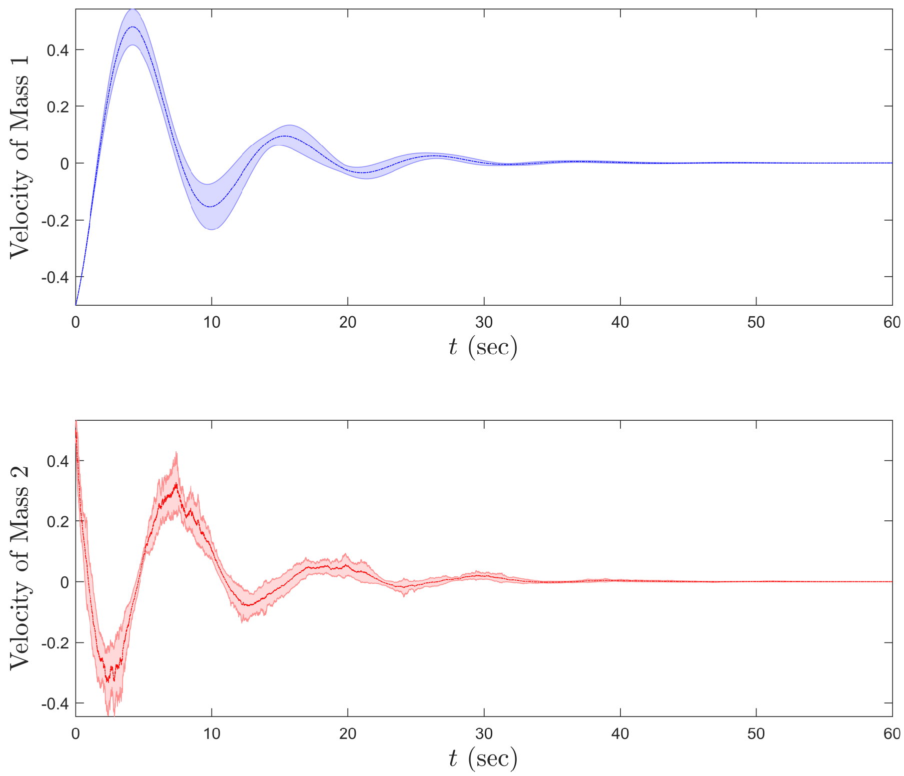


Figure 5.4.5: Mass Velocities versus time.

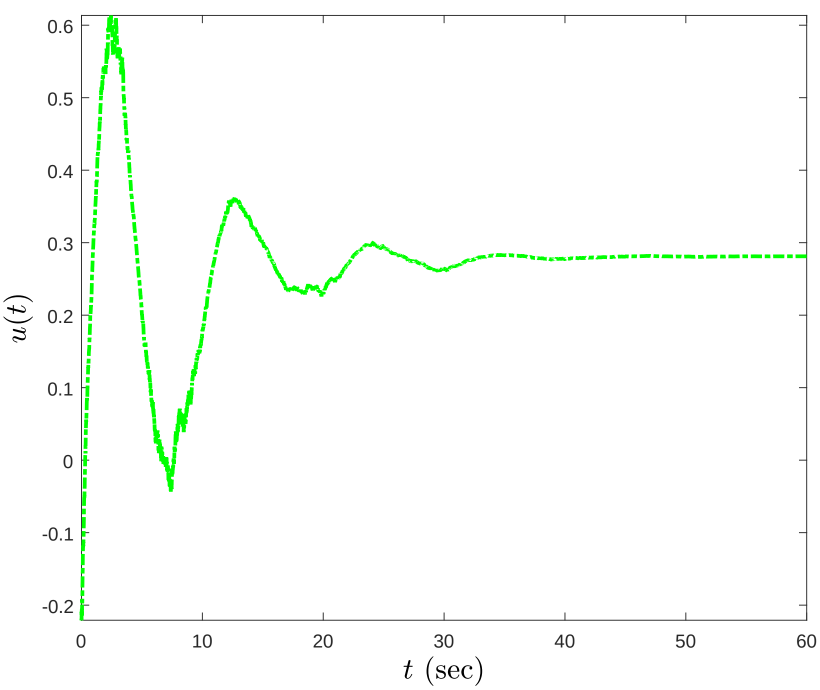


Figure 5.4.6: Control signal versus time.

Example 5.3. Finally, we consider the inverted pendulum of Example 5.1 with damping controlled via a dynamic controller. Specifically, consider the stochastic dynamical system

$$\begin{bmatrix} \mathrm{d}x_1(t) \\ \mathrm{d}x_2(t) \end{bmatrix} = \left(\begin{bmatrix} x_2(t) \\ g\sin(x_1(t)) - bx_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \right) \mathrm{d}t + \begin{bmatrix} 0 \\ \sigma x_2(t) \end{bmatrix} \mathrm{d}w(t),$$
$$[x_1(t_0),\ x_2(t_0)]^{\mathrm{T}} \overset{\text{a.s.}}{=} [x_{10},\ x_{20}]^{\mathrm{T}}, \quad t \geq t_0, \tag{5.42}$$

with problem data as in Example 5.1 and damping coefficient $b = 3\ \mathrm{N \cdot m \cdot s/rad}$. Note that (5.42) can be written in the form of (5.1) with $J(x)$, $G(x)$, $D(x)$ given by (5.37),

$$R(x) = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}, \tag{5.43}$$

$\mathcal{D} = \mathbb{R}^2$, and Hamiltonian function $H(\cdot)$ corresponding to the total energy in the system given by $H(x) = \frac{x_2^2}{2} + g\cos x_1$.

Here we design a reduced-order dynamic controller so that $x_{\mathrm{c}} \in \mathbb{R}$. The energy-Casimir function is chosen as $E(x, x_{\mathrm{c}}) = F_1(x_{\mathrm{c}}) - F_2(x)$, where $F_1(x_{\mathrm{c}}) = x_{\mathrm{c}}$ and $F_2(x) = x_1$, and the Hamiltonian function $H_{\mathrm{c}}(x_{\mathrm{c}})$ is chosen as $H_{\mathrm{c}}(x_{\mathrm{c}}) = -2g\cos x_{\mathrm{c}}$. Finally, we set

$$J_{\mathrm{c}}(x_{\mathrm{c}}) = 0, \quad R_{\mathrm{c}}(x_{\mathrm{c}}) = 0, \quad G_{\mathrm{c}}(x_{\mathrm{c}}) = 1, \quad D_{\mathrm{c}}(x_{\mathrm{c}}) = 0, \tag{5.44}$$

with $x_{\mathrm{c}}(0) \overset{\text{a.s.}}{=} x_1(0)$. In this case, it can be shown that (5.25)–(5.30) hold.

Next, to stabilize the pendulum at the equilibrium point $x_{\mathrm{e}} = [30^o, 0]^{\mathrm{T}}$ we assign the shaped Hamiltonian function $H_{\mathrm{s}}(x) = \frac{x_2^2}{2} - g\cos x_1$ for the closed-loop system. Now, it can be shown that (5.33)–(5.35) hold and $u = -y_{\mathrm{c}} = -2g\sin x_{\mathrm{c}} = -2g\sin x_1$. Next, note that $\mathcal{L}H_{\mathrm{s}}(x) = -x_2^2 \leq 0$, $x \in \mathcal{D}$. Hence, $\{x \in \mathcal{D} : \frac{\partial H_{\mathrm{s}}}{\partial x}(x)R(x)\left(\frac{\partial H_{\mathrm{s}}}{\partial x}(x)\right)^{\mathrm{T}} - \frac{1}{2}\mathrm{tr}\left(D^{\mathrm{T}}(x)\frac{\partial^2 H_{\mathrm{s}}}{\partial x^2}(x)D(x)\right) = 0\} = \{x \in \mathcal{D} : x_2 = 0\}$. Finally, since, for every $x \in \mathcal{D}$, $\mathrm{d}x_2 \neq 0$ if and only $x_1 \neq 0$, it follows that $\{x \in \mathcal{D} : x_2 = 0\} = \{x_{\mathrm{e}}\}$, and hence, the equilibrium solution $x(t) \equiv [30^o, 0]^{\mathrm{T}}$ is asymptotically stable in probability. Figure 5.4.7 shows the sample average along with the standard deviation of the angular displacement, angular velocity, and control profile versus time for $x(0) \overset{\text{a.s.}}{=} [0.1745,\ -5]^{\mathrm{T}}$ for 10 sample paths. $\triangle$

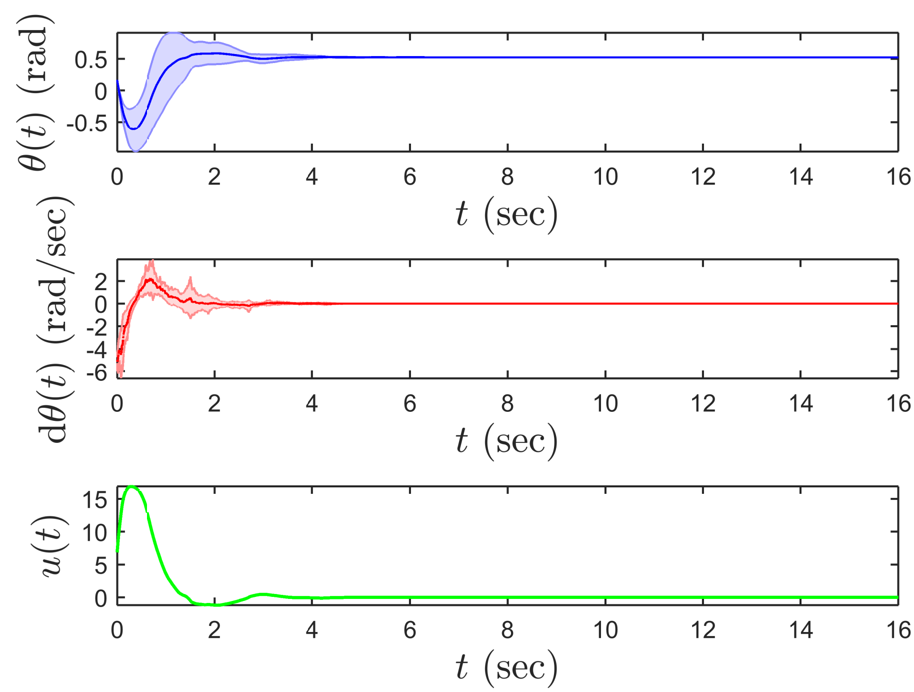


Figure 5.4.7: Angular displacement, angular velocity, and control profile versus time.

Chapter 6

Implications of Dissipativity, Inverse Optimal Control, and Stability Margins for Nonlinear Stochastic Feedback Regulators

6.1. Introduction

In a recent paper [115], we presented a framework for analyzing and designing feedback controllers for nonlinear stochastic dynamical systems. Specifically, a stochastic feedback control problem over an infinite horizon involving a nonlinear-nonquadratic performance functional was considered and the performance functional was evaluated in closed form as long as the nonlinear-nonquadratic cost functional considered was related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability in probability of the nonlinear closed-loop system. Furthermore, the Lyapunov function was shown to be the solution of the steady-state stochastic Hamilton-Jacobi-Bellman equation. The overall framework provides the foundation for extending stochastic linear-quadratic control to nonlinear-nonquadratic problems.

The approach in [115] focuses on the role of the Lyapunov function guaranteeing stochastic stability of the closed-loop system and its connection to the steady-state solution of the stochastic Hamilton-Jacobi-Bellman equation characterizing the optimal nonlinear feedback controller. In order to avoid the complexity in solving the stochastic steady-state, Hamilton-Jacobi-Bellman equation we do not attempt to minimize a given *given* cost functional, but rather, we parameterize a family of stochastically stabilizing controllers that minimizes a *derived* cost functional that provides the flexibility in specifying the control law. This corresponds to addressing an *inverse optimal stochastic control problem* [30, 39, 40, 62, 63, 91, 93, 115, 126].

The inverse optimal control design approach provides a framework for constructing the Lyapunov function for the closed-loop system that serves as an optimal value function and, as shown in [39, 126] for deterministic systems, achieves desired stability margins. Specifically, nonlinear inverse optimal controllers that minimize a *meaningful* (in the terminology of [39, 126]) nonlinear-nonquadratic performance criterion involving a nonlinear-nonquadratic, nonnegative-definite function of the state and a quadratic positive-definite function of the feedback control are shown to possess sector margin guarantees to component decoupled input nonlinearities in the conic sector $(\frac{1}{2}, \infty)$.

Using the framework developed in [115], in this chapter we derive stability margins for optimal and inverse optimal nonlinear *stochastic* feedback regulators. Specifically, sufficient conditions for gain, sector, and disk margin guarantees are obtained for nonlinear stochastic dynamical systems controlled by nonlinear optimal and inverse optimal Hamilton-Jacobi-Bellman controllers that minimize a nonlinear-nonquadratic performance criterion *with* cross-weighting terms. In the case where the cross-weighting term in the performance criterion is deleted our results recover the gain, sector, and disk margins for the deterministic optimal control problem presented in [126].

Alternatively, retaining the cross-terms in the performance criterion and specializing the optimal nonlinear-nonquadratic problem to a stochastic linear-quadratic problem with a multiplicative noise disturbance, our results recover the analogous gain and phase margins for the deterministic linear-quadratic optimal control problem given in [24]. Even though the inclusion of cross-weighting terms in the performance criterion is shown to degrade gain, sector, and disk margins, the extra flexibility provided by the cross-weighting terms makes it possible to guarantee optimal and inverse optimal nonlinear controllers that may be far superior in terms of transient performance over meaningful inverse optimal controllers.

Finally, using the newly developed notion of stochastic dissipativity [114] for controlled Markov diffusion processes characterized via extended Kalman-Yakubovich-Popov conditions in terms of the drift and diffusion dynamics developed in [114], we provide explicit connections between stochastic stability margins, stochastic meaningful inverse optimality, and stochastic dissipativity with respect to a specific quadratic supply rate. In particular, we derive a stochastic counterpart to the classical return difference inequality for continuous-time systems with continuously differentiable flows [22, 93] for stochastic dynamical systems and provide connections between stochastic dissipativity and optimality for stochastic nonlinear controllers. In particular, we show an equivalence between stochastic dissipativity and optimality holds for stochastic dynamical systems. Specifically, we show that an optimal nonlinear feedback controller $\phi(x)$ satisfying a return difference condition predicated on the infinitesimal generator of a controlled Markov diffusion process is equivalent to the fact that the stochastic dynamical system with input u and output $y = -\phi(x)$ is stochastically dissipative with respect to a supply rate of the form $[u+y]^{\mathrm{T}}[u+y] - u^{\mathrm{T}}u$.

6.2. Dissipativity Theory for Stochastic Systems

In this section, we present several key results on stochastic dissipativity developed in [114] that are necessary for the main results of this paper. Specifically, we consider nonlinear stochastic dynamical systems $\mathcal{G}$ of the form

$$\mathrm{d}x(t) = F(x(t), u(t))\mathrm{d}t + D(x(t), u(t))\mathrm{d}w(t), \quad x(t_0) \overset{\text{a.s.}}{=} x_0, \quad t \geq t_0, \tag{6.1}$$

$$y(t) = H(x(t), u(t)), \tag{6.2}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in \mathcal{H}_m^U$, $U \subseteq \mathbb{R}^m$, $y(t) \in \mathcal{H}_l^Y$, $Y \subseteq \mathbb{R}^l$, $F : \mathcal{D} \times U \to \mathbb{R}^n$, $D : \mathcal{D} \times U \to \mathbb{R}^{n \times d}$, and $H : \mathcal{D} \times U \to Y$. For the dynamical system $\mathcal{G}$ given by (6.1) and (6.2) defined on the state space $\mathcal{H}_n^{\mathcal{D}}$, $\mathcal{U}$ and $\mathcal{Y}$ define an input and output space, respectively, consisting of measurable bounded $\mathcal{H}_m^U$-valued and $\mathcal{H}_l^Y$-valued stochastic processes on the semi-infinite interval $[0, \infty)$. The set $\mathcal{H}_m^U$ contains the set of input values with measurable sample paths satisfying a *nonanticipativity condition*, that is, for every $u(\cdot) \in \mathcal{U}$ and $t \in [t_0, \infty)$, $u(t) \in \mathcal{H}_m^U$, and for all $t \geq s$, $w(t) - w(s)$ is independent of $u(\tau), w(\tau), \tau \leq s$, and $x(t_0)$. The set $\mathcal{H}_l^Y$ contains the set of output values, that is, for every $y(\cdot) \in \mathcal{Y}$ and $t \in [0, \infty)$, $y(t) \in \mathcal{H}_l^Y$. The spaces $\mathcal{U}$ and $\mathcal{Y}$ are assumed to be closed under the shift operator, that is, if $u(\cdot) \in \mathcal{U}$ (respectively, $y(\cdot) \in \mathcal{Y}$), then the function defined by $u_T \triangleq u(t+T)$ (respectively, $y_T \triangleq y(t+T)$) is contained in $\mathcal{U}$ (respectively, $\mathcal{Y}$) for all $T \geq 0$.

Furthermore, for the nonlinear stochastic dynamical system $\mathcal{G}$ we assume that the conditions for existence and uniqueness of solutions are satisfied, that is, $u(\cdot)$ satisfies sufficient regularity conditions such that the system (6.1) has a unique solution forward in time. Specifically, we assume that the control process $u(\cdot)$ in (6.1) is restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ adapted to the filtration $\{\mathcal{F}_t\}_{t \geq t_0}$ such that $u(\cdot)$ is nonanticipative and takes values in a compact, metrizable set $\mathcal{U}$. Furthermore, we assume the uniform Lipschitz continuity and growth conditions (3.3) and (3.4) hold for the controlled drift and diffusion terms $F(x, u)$ and $D(x, u)$ uniformly in u. In this case, it follows from Theorem 2.2.4 of [4] that there exists a pathwise unique solution to (6.1) in $(\Omega, \{\mathcal{F}_{t \geq t_0}\}, \mathbb{P}^{x_0})$.

For the stochastic dynamical system $\mathcal{G}$ given by (6.1) and (6.2), a function $r : \mathcal{H}_m^U \times \mathcal{H}_l^Y \to \mathcal{H}_1$ such that $r(0, 0) \overset{\text{a.s.}}{=} 0$ is called a *supply rate* if $r(u(t), y(t))$ is locally Lebesgue integrable for all input-output pairs satisfying (6.1) and (6.2), that is, for all input-output pairs $u(\cdot) \in \mathcal{U}$ and $y(\cdot) \in \mathcal{Y}$ satisfying (6.1) and (6.2), $r(\cdot, \cdot)$ satisfies $\mathbb{E}\left[\int_{t_1}^{t_2} |r(u(s), y(s))| \mathrm{d}s\right] < \infty$, $t_1, t_2 \geq 0$.

Definition 6.1. A stochastic dynamical system $\mathcal{G}$ of the form (6.1) and (6.2) is *stochastically dissipative with respect to the supply rate* $r(u, y)$ if there exists a measurable and nonnegative function $V_{\mathrm{s}} : \mathcal{D} \to \mathbb{R}$, called a *storage function* for $\mathcal{G}$, such that $V_{\mathrm{s}}(0) = 0$ and

$V_s(x(t)) - \int_{t_0}^{t} r(u(s), y(s))\mathrm{d}s$, $t \geq t_0$, is a $\mathcal{F}_t$-supermartingale for all $t_0, t \geq 0$, where $x(t)$, $t \geq t_0$, is the solution of (6.1) with $u(\cdot) \in \mathcal{U}$; or, equivalently,

$$\mathbb{E}\left[V_s(x(t))|\mathcal{F}_{\tau_0}\right] \leq V_s(x(\tau_0)) + \mathbb{E}\left[\int_{\tau_0}^{\tau} r(u(s), y(s))\mathrm{d}s\Big|\mathcal{F}_{\tau_0}\right], \quad \tau \overset{\text{a.s.}}{\geq} \tau_0, \tag{6.3}$$

where τ and τ_0 are finite $\mathcal{F}_t$-stopping times.

Definition 6.2. A nonlinear stochastic dynamical system $\mathcal{G}$ is *completely stochastically reachable* if, for all $x_0 \in \mathcal{D} \subseteq \mathbb{R}^n$ and $\varepsilon > 0$, there exist a finite random variable $\tau_{\mathcal{B}_\varepsilon(x_0)} \overset{\text{a.s.}}{\geq} 0$, called the *first hitting time*, defined by $\tau_{\mathcal{B}_\varepsilon(x_0)}(\omega) \triangleq \inf\{t \geq 0 : x(t, \omega) \in \mathcal{B}_\varepsilon(x_0)\}$, and a square integrable input $u(t)$ defined on $[0, \tau_{\mathcal{B}_\varepsilon(x_0)}]$ such that the state $x(t)$, $t \geq 0$, can be driven from $x(0) \overset{\text{a.s.}}{=} 0$ to $x(\tau_{\mathcal{B}_\varepsilon(x_0)})$ and $\mathbb{E}\left[\tau_{x_0}\right] < \infty$, where $\tau_{x_0} \triangleq \sup_{\varepsilon > 0} \tau_{\mathcal{B}_\varepsilon(x_0)}$ and the supremum is taken pointwise. A nonlinear stochastic dynamical system $\mathcal{G}$ is *zero-state observable* if $u(t) \overset{\text{a.s.}}{\equiv} 0$ and $y(t) \overset{\text{a.s.}}{\equiv} 0$ implies $x(t) \overset{\text{a.s.}}{\equiv} 0$.

If $V_s(\cdot)$ is two-times continuously differentiable, then an equivalent statement for the stochastic dissipativeness of $\mathcal{G}$ with respect to the supply rate $r(u, y)$ can be characterized by the infinitesimal generator $\mathcal{L}$.

Proposition 6.1 [114]. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (6.1) and (6.2). If $V_s : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ is two-times continuously differentiable and has a compact support, then $\mathcal{G}$ is stochastically dissipative with respect to supply rate $r(\cdot, \cdot)$ if and only if

$$\begin{aligned}\mathcal{L}V_s(x) &\triangleq \frac{\partial V(x)}{\partial x}F(x, u) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x, u)\frac{\partial^2 V(x)}{\partial x^2}D(x, u) \\ &\leq r(u, H(x, u)), \quad (x, u) \in \mathcal{D} \times U.\end{aligned} \tag{6.4}$$

Next, we show that stochastic dissipativeness of nonlinear affine stochastic dynamical systems $\mathcal{G}$ of the form

$$\mathrm{d}x(t) = \left[f(x(t)) + G(x(t))u(t)\right]\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(t_0) \overset{\text{a.s.}}{=} x_0, \quad t \geq t_0, \tag{6.5}$$

$$y(t) = h(x(t)) + J(x(t))u(t), \tag{6.6}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in \mathcal{H}_m^U$, $U \subseteq \mathbb{R}^m$ is an open set with $0 \in U$, $y(t) \in \mathcal{H}_l^Y$, $Y \subseteq \mathbb{R}^l$, $f : \mathcal{D} \to \mathbb{R}^n$, $G : \mathcal{D} \to \mathbb{R}^{n\times m}$, $D : \mathcal{D} \to \mathbb{R}^{n\times d}$, $h : \mathcal{D} \to \mathbb{R}^l$, and $J : \mathcal{D} \to \mathbb{R}^{l\times m}$, can be characterized in terms of the system functions $f(\cdot)$, $G(\cdot)$, $D(\cdot)$, $h(\cdot)$, and $J(\cdot)$. We assume that $f(\cdot)$, $G(\cdot)$, $D(\cdot)$, $h(\cdot)$, and $J(\cdot)$ are continuously differentiable mappings and $\mathcal{G}$ has at least one equilibrium so that, without loss of generality, $f(0) = 0$, $D(0) = 0$, and $h(0) = 0$. Furthermore, for the nonlinear stochastic dynamical system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions in forward time are satisfied.

For the following result we consider the special case of dissipative systems with quadratic supply rates [55–57, 149]. Specifically, we set $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, $Y = \mathbb{R}^l$, let $Q \in \mathbb{S}^l$, $R \in \mathbb{S}^m$, and $S \in \mathbb{R}^{l\times m}$ be given, where $\mathbb{S}^q$ denotes the set of $q \times q$ symmetric matrices, and assume $r(u, y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$. Furthermore, we assume that there exists a function $\kappa : \mathbb{R}^l \to \mathbb{R}^m$ such that $\kappa(0) = 0$ and $r(\kappa(y), y) < 0$, $y \neq 0$, so that, as shown by Theorem 3.2 of [114], all storage functions for $\mathcal{G}$ are positive definite. Moreover, we assume that there exists a two-times continuously differentiable storage function $V_{\mathrm{s}}(x)$, $x \in \mathbb{R}^n$, for the stochastic dynamical system $\mathcal{G}$.

Theorem 6.1 [114]. Let $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, $R \in \mathbb{S}^m$, and let $\mathcal{G}$ be zero-state observable and completely stochastically reachable. $\mathcal{G}$ is stochastically dissipative with respect to the quadratic supply rate $r(u, y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$ if and only if there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that $V_{\mathrm{s}}(\cdot)$ is two-times continuously differentiable and positive definite, $V_{\mathrm{s}}(0) = 0$, and, for all $x \in \mathbb{R}^n$,

$$0 = V_{\mathrm{s}}'(x)f(x) + \tfrac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V_{\mathrm{s}}''(x)D(x) - h^{\mathrm{T}}(x)Qh(x) + \ell^{\mathrm{T}}(x)\ell(x), \tag{6.7}$$

$$0 = \tfrac{1}{2}V_{\mathrm{s}}'(x)G(x) - h^{\mathrm{T}}(x)(QJ(x) + S) + \ell^{\mathrm{T}}(x)\mathcal{W}(x), \tag{6.8}$$

$$0 = R + S^{\mathrm{T}}J(x) + J^{\mathrm{T}}(x)S + J^{\mathrm{T}}(x)QJ(x) - \mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x). \tag{6.9}$$

If, alternatively,

$$\mathcal{N}(x) \triangleq R + S^{\mathrm{T}}J(x) + J^{\mathrm{T}}(x)S + J^{\mathrm{T}}(x)QJ(x) > 0, \quad x \in \mathbb{R}^n,$$

then $\mathcal{G}$ is stochastically dissipative with respect to the quadratic supply rate $r(u, y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$ if and only if there exists a two-times continuously differentiable function $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$ such that $V_{\mathrm{s}}(\cdot)$ is positive definite, $V_{\mathrm{s}}(0) = 0$, and, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} 0 \geq V_{\mathrm{s}}'(x)f(x) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V_{\mathrm{s}}''(x)D(x) - h^{\mathrm{T}}(x)Qh(x) + [\tfrac{1}{2}V_{\mathrm{s}}'(x)G(x) - h^{\mathrm{T}}(x)(QJ(x) + S)] \\ \cdot \mathcal{N}^{-1}(x)[\tfrac{1}{2}V_{\mathrm{s}}'(x)G(x) - h^{\mathrm{T}}(x)(QJ(x) + S)]^{\mathrm{T}}. \qquad (6.10) \end{aligned}$$

6.3. Stability Margins for Stochastic Feedback Regulators

To develop relative stability margins for nonlinear stochastic regulators consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))u(t)]\,\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \qquad (6.11)$$

$$y(t) = -\phi(x(t)), \qquad (6.12)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $f(0) = 0$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $D : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ satisfies $D(0) = 0$, and $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is an admissible feedback controller such that $\mathcal{G}$ is globally asymptotically stable in probability with $u = -y$. Furthermore, we assume that $\mathcal{G}$ is zero-state observable.

Next, we define the relative stability margins for $\mathcal{G}$ given by (6.11) and (6.12). Specifically, let $u_{\mathrm{c}} \triangleq -y$, $y_{\mathrm{c}} \triangleq u$, and consider the negative feedback interconnection $u = \Delta(-y)$ of $\mathcal{G}$ and $\Delta(\cdot)$ given in Figure 6.3.1, where $\Delta(\cdot)$ is either a linear operator $\Delta(u_{\mathrm{c}}) = \Delta u_{\mathrm{c}}$, a nonlinear static operator $\Delta(u_{\mathrm{c}}) = \sigma(u_{\mathrm{c}})$, or a nonlinear dynamic operator $\Delta(\cdot)$ with input u_{c} and output y_{c}. Furthermore, we assume that in the nominal case $\Delta(\cdot) = I(\cdot)$ so that the nominal closed-loop system is globally asymptotically stable in probability.

Definition 6.3. Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha \leq 1 \leq \beta < \infty$. Then the nonlinear stochastic dynamical system $\mathcal{G}$ given by (6.11) and (6.12) is said to have a *gain margin* (α, β)

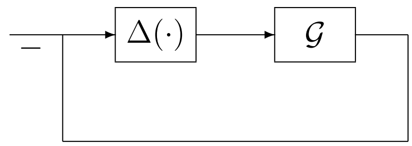


Figure 6.3.1: Multiplicative input uncertainty of $\mathcal{G}$ and input operator $\Delta(\cdot)$.

if the negative feedback interconnection of $\mathcal{G}$ and $\Delta(u_{\rm c}) = \Delta u_{\rm c}$ is globally asymptotically stable in probability for all $\Delta = \mathrm{diag}[k_1, \ldots, k_m]$, where $k_i \in (\alpha, \beta)$, $i = 1, \ldots, m$.

Definition 6.4. Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha \le 1 \le \beta < \infty$. Then the nonlinear stochastic dynamical system $\mathcal{G}$ given by (6.11) and (6.12) is said to have a *sector margin* (α, β) if the negative feedback interconnection of $\mathcal{G}$ and $\Delta(u_{\rm c}) = \sigma(u_{\rm c})$ is globally asymptotically stable in probability for all nonlinearities $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ such that $\sigma(0) = 0$, $\sigma(u_{\rm c}) = [\sigma_1(u_{{\rm c}1}), \ldots, \sigma_m(u_{{\rm c}m})]^{\rm T}$, and $\alpha u_{ci}^2 < \sigma_i(u_{{\rm c}i}) u_{{\rm c}i} < \beta u_{{\rm c}i}^2$, for all $u_{{\rm c}i} \neq 0$, $i = 1, \ldots, m$.

Definition 6.5. Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha \le 1 \le \beta < \infty$. Then the nonlinear stochastic dynamical system $\mathcal{G}$ given by (6.11) and (6.12) is said to have a *disk margin* (α, β) if the negative feedback interconnection of $\mathcal{G}$ and $\Delta(\cdot)$ is globally asymptotically stable in probability for all dynamic operators $\Delta(\cdot)$ such that $\Delta(\cdot)$ is zero-state observable and stochastically dissipative with respect to the supply rate $r(u_{\rm c}, y_{\rm c}) = u_{\rm c}^{\rm T} y_{\rm c} - \frac{1}{\hat{\alpha}+\hat{\beta}} y_{\rm c}^{\rm T} y_{\rm c} - \frac{\hat{\alpha}\hat{\beta}}{\hat{\alpha}+\hat{\beta}} u_{\rm c}^{\rm T} u_{\rm c}$, where $\hat{\alpha} = \alpha + \delta$, $\hat{\beta} = \beta - \delta$, and $\delta \in \mathbb{R}$ such that $0 < 2\delta < \beta - \alpha$.

Definition 6.6. Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha \le 1 \le \beta < \infty$. Then the nonlinear stochastic dynamical system $\mathcal{G}$ given by (6.11) and (6.12) is said to have a *structured disk margin* (α, β) if the negative feedback interconnection of $\mathcal{G}$ and $\Delta(\cdot)$ is globally asymptotically stable in probability for all dynamic operators $\Delta(\cdot)$ such that $\Delta(\cdot)$ is zero-state observable, $\Delta(u_{\rm c}) = \mathrm{diag}[\delta_1(u_{{\rm c}1}), \ldots, \delta_m(u_{{\rm c}m})]$, and $\delta_i(\cdot)$, $i = 1, \ldots, m$, is stochastically dissipative with respect to the supply rate $r(u_{{\rm c}i}, y_{{\rm c}i}) = u_{{\rm c}i} y_{{\rm c}i} - \frac{1}{\hat{\alpha}+\hat{\beta}} y_{{\rm c}i}^2 - \frac{\hat{\alpha}\hat{\beta}}{\hat{\alpha}+\hat{\beta}} u_{{\rm c}i}^2$, where $\hat{\alpha} = \alpha + \delta$, $\hat{\beta} = \beta - \delta$, and $\delta \in \mathbb{R}$ such that $0 < 2\delta < \beta - \alpha$.

Remark 6.1. Note that if $\mathcal{G}$ has a disk margin (α, β), then $\mathcal{G}$ has gain and sector margins (α, β).

6.4. Nonlinear-Nonquadratic Optimal Regulators for Stochastic Dynamical Systems

In this section, we consider a control problem involving a notion of optimality with respect to a nonlinear-nonquadratic cost functional. In particular, consider the controlled nonlinear stochastic dynamical system (6.1), where $u(\cdot)$ is restricted to the class of admissible controls consisting of measurable functions $u(\cdot)$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ such that $u(t) \in \mathcal{H}_m^U$ for almost all $t \geq t_0$ and $u(\cdot)$ is nonanticipative and takes values in a given compact, metrizable set $\mathcal{U}$.

A measurable function $\phi : \mathcal{D} \to U$ satisfying $\phi(0) = 0$ is called a *control law*. If $u(t) = \phi(x(t))$, $t \geq t_0$, where $\phi(\cdot)$ is a control law and $x(t)$, $t \geq t_0$, satisfies (6.1), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot)$ has values in U. Given a control law $\phi(\cdot)$ and a feedback control law $u(t) = \phi(x(t))$, $t \geq t_0$, the *closed-loop system* (6.1) has the form

$$\mathrm{d}x(t) = F(x(t), \phi(x(t)))\mathrm{d}t + D(x(t), \phi(x(t)))\mathrm{d}w(t) \quad x(t_0) \overset{\text{a.s.}}{=} x_0, \quad t \geq t_0. \tag{6.13}$$

Next, we present a main theorem for stochastic stabilization characterizing feedback controllers that guarantee local and global closed-loop stability in probability and minimize a nonlinear-nonquadratic performance measure. For the statement of this result, let $L : \mathcal{D} \times U \to \mathbb{R}$ be jointly continuous in x and u, and, for every $\rho \in (0, 1)$, define the set of stochastic regulation controllers given by

$$\mathcal{S}(x_0, \rho) \triangleq \Big\{u(\cdot) : u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (6.1) is such that } \mathbb{P}^{x_0}\Big(\mathfrak{B}_{x_0}^{u(\cdot)}\Big) \geq 1 - \rho,$$
$$\text{where } \mathfrak{B}_{x_0}^{u(\cdot)} \triangleq \Big\{x(\{t \geq t_0\}, \omega) : \lim_{t\to\infty} \|x(t, \omega)\| = 0, \, \omega \in \Omega\Big\}\Big\}.$$

Furthermore, define the indicator function of the set $\mathfrak{B}_{x_0}^{u(\cdot)}$ by

$$\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \triangleq \begin{cases} 1, & \text{if } x(\{t \geq t_0\}, \omega) \in \mathfrak{B}_{x_0}^{u(\cdot)}, \\ 0, & \text{otherwise.} \end{cases}$$

The set $\mathfrak{B}_{x_0}^{u(\cdot)}$ denotes the set of all controlled sample paths of (15) for which $\lim_{t\to\infty} \|x(t,\omega)\| = 0$ and $x(\{t \geq t_0\}, \omega) \in \mathfrak{B}_{x_0}^{u(\cdot)}$, $\omega \in \Omega$. Since in *local* stochastic stability theory there exists a probability of less than or equal to ρ that the system solution $x(t,\omega)$ leaves the subset $\mathcal{B}_\varepsilon(0)$ for every $x_0 \in \mathcal{B}_\delta(0)$, that is, the probability of escape is continuous at $x_0 = 0$ with small deviations from the equilibrium implying a small probability of escape, the set $\mathfrak{B}_{x_0}^{u(\cdot)}$ as well as $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot)}\right)$ are necessary for defining a well-posed cost functional for the optimal control problem formulation given in Theorem 6.2.

Theorem 6.2 [115]. Consider the nonlinear stochastic controlled dynamical system (6.1) with performance measure

$$J\left(x_0, u(\cdot), \mathfrak{B}_{x_0}^{u(\cdot)}\right) \triangleq \frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot)}\right)} \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t), u(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t\right], \tag{6.14}$$

where $u(\cdot)$ is an admissible control and $\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)$ denotes the indicator function of the set $\mathfrak{B}_{x_0}^{u(\cdot)}$. Assume that there exist a two-times continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ and a control law $\phi : \mathcal{D} \to U$ such that

$$V(0) = 0, \tag{6.15}$$

$$V(x) > 0, \qquad x \in \mathcal{D}, \qquad x \neq 0, \tag{6.16}$$

$$\phi(0) = 0, \tag{6.17}$$

$$V'(x)F(x,\phi(x)) + \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x,\phi(x))V''(x)D(x,\phi(x)) < 0, \qquad x \in \mathcal{D}, \qquad x \neq 0, \tag{6.18}$$

$$H(x,\phi(x)) = 0, \qquad x \in \mathcal{D}, \tag{6.19}$$

$$H(x,u) \geq 0, \qquad x \in \mathcal{D}, \qquad u \in U, \tag{6.20}$$

where

$$H(x,u) \triangleq L(x,u) + V'(x)F(x,u) + \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x,u)V''(x)D(x,u). \tag{6.21}$$

Then, with the feedback control $u(\cdot) = \phi(x(\cdot))$, the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the closed-loop system (6.13) is locally asymptotically stable in probability and, for every $\rho \in (0, 1)$, there exist $\delta = \delta(\rho)$ and $\mathfrak{B}_{x_0}^{\phi(x(\cdot))}$ with $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(x(\cdot))}\right) \geq 1 - \rho$ such that, for all $x_0 \in \mathcal{B}_\delta(0) \subseteq \mathcal{D}$,

$$J\left(x_0, \phi(x(\cdot)), \mathfrak{B}_{x_0}^{\phi(x(\cdot))}\right) = V(x_0). \tag{6.22}$$

In addition, if $x_0 \in \mathcal{B}_\delta(0)$, then the feedback control $u(\cdot) = \phi(x(\cdot))$ minimizes (6.14) in the sense that

$$J\left(x_0, \phi(x(\cdot)), \mathfrak{B}_{x_0}^{\phi(x(\cdot))}\right) = \min_{u(\cdot)\in\mathcal{S}(x_0,\rho)} J\left(x_0, u(\cdot), \mathfrak{B}_{x_0}^{u(\cdot)}\right). \tag{6.23}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, and $V(x) \to \infty$ as $\|x\| \to \infty$, then the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the closed-loop system (6.13) is globally asymptotically stable in probability and (6.23) holds with $\rho = 0$ and $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(x(\cdot))}\right) = 1$, $x_0 \in \mathbb{R}^n$.

It is important to note here that in the case where the optimal feedback control $\phi(\cdot)$ guarantees global asymptotic stability in probability, $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(\cdot)}\right) = 1$, and hence, $\mathbb{1}_{\mathfrak{B}_{x_0}^{\phi(\cdot)}}(\omega) \overset{\text{a.s.}}{=} 1$. Moreover, all the admissible controls $u(\cdot)$ that guarantee global attraction in probability also satisfy $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot)}\right) = 1$ for all $x_0 \in \mathbb{R}^n$, and hence, $\rho = 0$ and $\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \overset{\text{a.s.}}{=} 1$. In this case,

$$\begin{aligned} J\left(x_0, u(\cdot), \mathfrak{B}_{x_0}^{u(\cdot)}\right) &= \frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot)}\right)} \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t), u(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \mathrm{d}t\right] \\ &= \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t), u(t)) \mathrm{d}t\right] \end{aligned} \tag{6.24}$$

and

$$\begin{aligned} J\left(x_0, \phi(\cdot), \mathfrak{B}_{x_0}^{\phi(\cdot)}\right) &= \frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(\cdot)}\right)} \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t), \phi(x(t))) \mathbb{1}_{\mathfrak{B}_{x_0}^{\phi(\cdot)}}(\omega) \mathrm{d}t\right] \\ &= \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t), \phi(x(t))) \mathrm{d}t\right]. \end{aligned} \tag{6.25}$$

Thus, in the remainder of the paper, we omit the dependence on $\mathfrak{B}_{x_0}^{\phi(\cdot)}$ and $\mathfrak{B}_{x_0}^{u(\cdot)}$ in the cost functional and we write $\mathcal{S}(x_0)$ for $\mathcal{S}(x_0, \rho)$ for all the results concerning globally stabilizing controllers in probability.

Next, we specialize Theorem 6.2 to affine stochastic dynamical systems. Specifically, we construct nonlinear feedback controllers using an optimal control framework that minimizes a nonlinear-nonquadratic performance criterion. This is accomplished by choosing the controller such that the infinitesimal generator is negative along the closed-loop system sample trajectories while providing sufficient conditions for the existence of stochastically asymptotically stabilizing solutions to the stochastic Hamilton-Jacobi-Bellman equation. Thus, these results provide a family of globally stochastically stabilizing controllers parameterized by the cost functional that is minimized.

The controllers obtained next are predicated on an inverse optimal stochastic control problem [30,39,40,62,63,91,93,115,126]. Consider the nonlinear affine stochastic dynamical system given by (6.11) with performance integrands $L(x,u)$ of the form

$$L(x,u) = L_1(x) + L_2(x)u + u^{\mathrm{T}}R_2(x)u, \tag{6.26}$$

where $L_1 : \mathbb{R}^n \to \mathbb{R}$, $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, and $R_2 : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, where $R_2(x) > 0$, $x \in \mathbb{R}^n$, so that

$$J(x_0, u(\cdot)) = \mathbb{E}^{x_0}\left[\int_0^\infty [L_1(x(t)) + L_2(x(t))u(t) + u^{\mathrm{T}}(t)R_2(x(t))u(t)]\mathrm{d}t\right]. \tag{6.27}$$

Theorem 6.3 [115]. Consider the nonlinear controlled affine stochastic dynamical system (6.11) with performance measure (6.27). Assume that there exist a two-times continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ and a function $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$ such that

$$V(0) = 0, \tag{6.28}$$

$$L_2(0) = 0, \tag{6.29}$$

$$V(x) > 0, \qquad x \in \mathbb{R}^n, \qquad x \neq 0, \tag{6.30}$$

$$V'(x)\left[f(x) - \tfrac{1}{2}G(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) - \tfrac{1}{2}G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)\right] + \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{6.31}$$

and $V(x) \to \infty$ as $\|x\| \to \infty$. Then the zero solution $x(t) \stackrel{\text{a.s.}}{\equiv} 0$ of the closed-loop system

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))\phi(x(t))]\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \stackrel{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{6.32}$$

is globally asymptotically stable in probability with the feedback control law

$$\phi(x) = -\tfrac{1}{2}R_2^{-1}(x)[V'(x)G(x) + L_2(x)]^{\mathrm{T}}, \tag{6.33}$$

and the performance measure (6.27), with

$$L_1(x) = \phi^{\mathrm{T}}(x)R_2(x)\phi(x) - V'(x)f(x) - \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x), \tag{6.34}$$

is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \qquad x_0 \in \mathbb{R}^n. \tag{6.35}$$

Finally,

$$J(x_0, \phi(x(\cdot))) = V(x_0), \qquad x_0 \in \mathbb{R}^n. \tag{6.36}$$

Note that (6.31) is equivalent to

$$\mathcal{L}V(x) \triangleq V'(x)[f(x) + G(x)\phi(x)] + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{6.37}$$

with $\phi(x)$ given by (6.33). Furthermore, conditions (6.28), (6.30), and (6.37) ensure that $V(\cdot)$ is a Lyapunov function for the closed-loop system (6.32). As discussed in [115], it is important to recognize that the function $L_2(x)$, which appears in the integrand of the performance measure (6.26), is an arbitrary function of $x \in \mathbb{R}^n$ subject to conditions (6.29) and (6.31). Thus, $L_2(x)$ provides flexibility in choosing the control law.

With $L_1(x)$ given by (6.34) and $\phi(x)$ given by (6.33), $L(x,u)$ can be expressed as

$$\begin{aligned} L(x,u) \ &= \ u^{\mathrm{T}}R_2(x)u - \phi^{\mathrm{T}}(x)R_2(x)\phi(x) + L_2(x)(u - \phi(x)) \\ &\qquad -V'(x)[f(x) + G(x)\phi(x)] - \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) \\ &= \ \left[u + \tfrac{1}{2}R_2^{-1}(x)L_2^{\mathrm{T}}(x)\right]^{\mathrm{T}}R_2(x)\left[u + \tfrac{1}{2}R_2^{-1}(x)L_2^{\mathrm{T}}(x)\right] - V'(x)[f(x) \end{aligned}$$

$$+G(x)\phi(x)] - \frac{1}{2}\text{tr } D^{\mathrm{T}}(x)V''(x)D(x) - \tfrac{1}{4}V'(x)G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x). \tag{6.38}$$

Since $R_2(x) > 0$, $x \in \mathbb{R}^n$, the first term on the right-hand side of (6.38) is nonnegative, while (6.37) implies that the second, third, and fourth terms collectively are nonnegative. Thus, it follows that

$$L(x,u) \geq -\tfrac{1}{4}V'(x)G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x), \tag{6.39}$$

which shows that $L(x,u)$ may be negative. As a result, there may exist a control input u for which the performance measure $J(x_0,u)$ is negative. However, if the control u is a regulation controller, that is, $u \in \mathcal{S}(x_0)$, then it follows from (6.35) and (6.36) that

$$J(x_0,u(\cdot)) \geq V(x_0) \geq 0, \qquad x_0 \in \mathbb{R}^n, \qquad u(\cdot) \in \mathcal{S}(x_0). \tag{6.40}$$

Furthermore, in this case, substituting $u = \phi(x)$ into (6.38) yields

$$L(x,\phi(x)) = -V'(x)[f(x) + G(x)\phi(x)] - \frac{1}{2}\text{tr } D^{\mathrm{T}}(x)V''(x)D(x), \tag{6.41}$$

which, by (6.37), is positive.

6.5. Gain, Sector, and Disk Margins of Nonlinear-Nonquadratic Optimal Regulators for Stochastic Dynamical Systems

In this section, we derive guaranteed gain, sector, and disk margins for nonlinear optimal and inverse optimal regulators that minimize a nonlinear-nonquadratic performance criterion for stochastic dynamical systems. Specifically, sufficient conditions that guarantee gain, sector, and disk margins are given in terms of the state, control, and cross-weighting nonlinear-nonquadratic weighting functions.

In particular, we consider the nonlinear stochastic dynamical system given by

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))u(t)]\,\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{6.42}$$

$$y(t) = -\phi(x(t)), \tag{6.43}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}^m$, with a nonlinear-nonquadratic performance criterion

$$J(x_0, u(\cdot)) = \mathbb{E}^{x_0} \left[\int_0^\infty [L_1(x(t)) + L_2(x(t))u(t) + u^{\mathrm{T}}(t)R_2(x(t))u(t)]\mathrm{d}t \right], \tag{6.44}$$

where $L_1 : \mathbb{R}^n \to \mathbb{R}$, $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, and $R_2 : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ are given such that $R_2(x) > 0$, $x \in \mathbb{R}^n$, and $L_2(0) = 0$. In this case, the optimal nonlinear feedback controller $u = \phi(x)$ that minimizes the nonlinear-nonquadratic performance criterion (6.44) is given by the following result.

Theorem 6.4. Consider the nonlinear stochastic dynamical system (6.42) and (6.43) with performance functional (6.44). Assume that there exists a two-times continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$V(0) = 0, \tag{6.45}$$

$$V(x) > 0, \qquad x \in \mathbb{R}^n, \qquad x \neq 0, \tag{6.46}$$

$$L_2(0) = 0, \tag{6.47}$$

$$\begin{aligned} V'(x)[f(x) - \tfrac{1}{2}G(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) - \tfrac{1}{2}G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)] \\ +\tfrac{1}{2}\mathrm{tr}\; D^{\mathrm{T}}(x)V''(x)D(x) < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \end{aligned} \tag{6.48}$$

$$\begin{aligned} 0 = L_1(x) + V'(x)f(x) + \tfrac{1}{2}\mathrm{tr}\; D^{\mathrm{T}}(x)V''(x)D(x) - \tfrac{1}{4}[V'(x)G(x) + L_2(x)] \\ \cdot R_2^{-1}(x)[V'(x)G(x) + L_2(x)]^{\mathrm{T}}, \quad x \in \mathbb{R}^n, \end{aligned} \tag{6.49}$$

and

$$V(x) \to \infty \text{ as } \|x\| \to \infty. \tag{6.50}$$

Then the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the closed-loop system

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))\phi(x(t))]\,\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{6.51}$$

is globally asymptotically stable in probability with the feedback control law

$$\phi(x) = -\tfrac{1}{2}R_2^{-1}(x)[V'(x)G(x) + L_2(x)]^{\mathrm{T}}, \tag{6.52}$$

and the performance functional (6.44) is minimized in the sense that

$$J(x_0,\phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0,u(\cdot)), \qquad x_0\in\mathbb{R}^n. \tag{6.53}$$

Finally,

$$J(x_0,\phi(x(\cdot))) = V(x_0), \qquad x_0\in\mathbb{R}^n. \tag{6.54}$$

Proof. The proof is identical to the proof of Theorem 6.3 given in [115]. $\square$

The following key lemma is needed for developing the main result of this section.

Lemma 6.1. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (6.11) and (6.12), where $\phi(x)$ is a stochastically stabilizing feedback control law given by (6.33) and where $V(x)$, $x\in\mathbb{R}^n$, satisfies

$$\begin{aligned} 0 = {} & V'(x)f(x) + L_1(x) - \tfrac{1}{4}[V'(x)G(x)+L_2(x)]R_2^{-1}(x)[V'(x)G(x)+L_2(x)]^{\mathrm{T}} \\ & + \tfrac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x). \end{aligned} \tag{6.55}$$

Furthermore, suppose there exists $\theta\in\mathbb{R}$ such that $0<\theta<1$ and

$$(1-\theta^2)L_1(x) - \tfrac{1}{4}L_2(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) \ge 0, \qquad x\in\mathbb{R}^n. \tag{6.56}$$

Then for all $u(\cdot)\in\mathcal{U}$ and $t_1,t_2\ge 0$, $t_1<t_2$, the solution $x(t)$, $t\ge 0$, to (6.11) and (6.12) satisfies

$$\mathcal{L}V(x) \le [u+y]^{\mathrm{T}}R_2(x)[u+y] - \theta^2 u^{\mathrm{T}}R_2(x)u, \tag{6.57}$$

which implies

$$\begin{aligned} \mathbb{E}[V(x(t_2))|\mathcal{F}_{t_1}] \le V(x(t_1)) + \mathbb{E}\bigg[\int_{t_1}^{t_2} & ([u(s)+y(s)]^{\mathrm{T}}R_2(x(s))[u(s)+y(s)] \\ & - \theta^2 u^{\mathrm{T}}(s)R_2(x(s))u(s))\mathrm{d}s\Big|\mathcal{F}_{t_1}\bigg], \end{aligned} \tag{6.58}$$

Proof. Note that it follows from (6.55) and (6.56) that for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$,

$$\begin{aligned}
\theta^2 u^{\mathrm{T}} R_2(x) u &\leq \theta^2 u^{\mathrm{T}} R_2(x) u + \left[\frac{1}{2\sqrt{1-\theta^2}} L_2(x) R_2^{-1}(x) + \sqrt{1-\theta^2} u^{\mathrm{T}}\right] \\
&\quad \cdot R_2(x) \left[\frac{1}{2\sqrt{1-\theta^2}} L_2(x) R_2^{-1}(x) + \sqrt{1-\theta^2} u^{\mathrm{T}}\right]^{\mathrm{T}} \\
&= u^{\mathrm{T}} R_2(x) u + \frac{1}{4(1-\theta^2)} L_2(x) R_2^{-1}(x) L_2^{\mathrm{T}}(x) + L_2(x) u \\
&\leq u^{\mathrm{T}} R_2(x) u + L_2(x) u + L_1(x) \\
&= u^{\mathrm{T}} R_2(x) u + L_2(x) u - V'(x) f(x) + \phi^{\mathrm{T}}(x) R_2(x) \phi(x) - \frac{1}{2} \mathrm{tr}\ D^{\mathrm{T}}(x) V''(x) D(x) \\
&= [u+y]^{\mathrm{T}} R_2(x) [u+y] - V'(x)[f(x) + G(x) u] - \frac{1}{2} \mathrm{tr}\ D^{\mathrm{T}}(x) V''(x) D(x),
\end{aligned}$$

which implies that, for all $u(\cdot) \in \mathcal{U}$,

$$\theta^2 u^{\mathrm{T}} R_2(x) u \leq [u+y]^{\mathrm{T}} R_2(x) [u+y] - \mathcal{L} V(x). \tag{6.59}$$

Now, using Dynkin's formula [94, Thm 7.12],

$$\begin{aligned}
\mathbb{E}[V(x(t_2)) | \mathcal{F}_{t_1}] \leq V(x(t_1)) + \mathbb{E}\Bigg[\int_{t_1}^{t_2} &([u(s)+y(s)]^{\mathrm{T}} R_2(x(s)) [u(s)+y(s)] \\
&- \theta^2 u^{\mathrm{T}}(s) R_2(x(s)) u(s)) \mathrm{d}s \Big| \mathcal{F}_{t_1}\Bigg],
\end{aligned} \tag{6.60}$$

is immediate. □

Next, we present disk margins for the nonlinear-nonquadratic optimal regulator given by Theorem 6.3. First, we consider the case in which $R_2(x)$, $x \in \mathbb{R}^n$, is a constant diagonal matrix.

Theorem 6.5. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (6.11) and (6.12), where $\phi(x)$ is a stochastically stabilizing feedback control law given by (6.33) and where $V(x)$, $x \in \mathbb{R}^n$, satisfies (6.34). If $R_2(x) \equiv \mathrm{diag}[r_1, \ldots, r_m]$, where $r_i > 0$, $i = 1, \ldots, m$, and there exists $\theta \in \mathbb{R}$ such that $0 < \theta < 1$ and (6.56) is satisfied, then the nonlinear stochastic dynamical system $\mathcal{G}$ has a structured disk margin $(\frac{1}{1+\theta}, \frac{1}{1-\theta})$. If, in addition, $R_2(x) \equiv I$ and there exists $\theta \in \mathbb{R}$ such that $0 < \theta < 1$ and (6.56) is satisfied, then the nonlinear stochastic dynamical system $\mathcal{G}$ has a disk margin $(\frac{1}{1+\theta}, \frac{1}{1-\theta})$.

Proof. Note that for all $u(\cdot) \in \mathcal{U}$, it follows from Lemma 6.1 that the solution $x(t)$, $t \geq 0$, to (6.11) satisfies

$$\mathcal{L}V(x) \leq [u+y]^{\mathrm{T}} R_2 [u+y] - \theta^2 u^{\mathrm{T}} R_2 u. \tag{6.61}$$

Hence, with the storage function $V_{\mathrm{s}}(x) = \frac{1}{2}V(x)$, it follows from Proposition 6.1 that $\mathcal{G}$ is stochastically dissipative with respect to the supply rate $r(u,y) = u^{\mathrm{T}} R_2 y + \frac{1-\theta^2}{2} u^{\mathrm{T}} R_2 u + y^{\mathrm{T}} R_2 y$. Now, the result is a direct consequence of Definitions 6.6 and 6.5 with $\alpha = \frac{1}{1+\theta}$ and $\beta = \frac{1}{1-\theta}$. □

Example 6.1. Consider the nonlinear stochastic dynamical system given by

$$\mathrm{d}x_1(t) = -x_1(t) + x_1(t)x_2^2(t) + g_1 x_1(t)\mathrm{d}w(t), \quad x_1(0) \overset{\text{a.s.}}{=} x_{10}, \quad t \geq 0, \tag{6.62}$$

$$\mathrm{d}x_2(t) = -x_2(t) + x_1(t)u(t) + g_2 x_2(t)\mathrm{d}w(t), \quad x_2(0) \overset{\text{a.s.}}{=} x_{20}, \tag{6.63}$$

where $g_1 < \sqrt{2}$ and $g_2 < \sqrt{2}$, with performance functional

$$J(x_{10}, x_{20}, u(\cdot)) = \mathbb{E}^{x_0}\left[\int_0^\infty [(2-g_1^2)x_1^2(t) + (2-g_2^2)x_2^2(t) + \tfrac{1}{2}u^2(t)]\mathrm{d}t\right]. \tag{6.64}$$

To design an optimal control law $\phi(x_1, x_2)$ that minimizes (6.64) we use Theorem 6.4 with $x = [x_1,\ x_2]^{\mathrm{T}}$, $f(x) = [-x_1 + x_1 x_2^2,\ -x_2]^{\mathrm{T}}$, $G(x) = [0,\ x_1]^{\mathrm{T}}$, $D(x) = [g_1 x_1,\ g_2 x_2]^{\mathrm{T}}$, $L_1(x) = (2-g_1^2)x_1^2 + (2-g_2^2)x_2^2$, $L_2(x) = 0$, and $R_2(x) = \frac{1}{2}$. In particular, it follows from (6.49) that

$$\begin{aligned} 0 = V'(x)\begin{bmatrix} -x_1 + x_1 x_2^2 \\ -x_2 \end{bmatrix} - \tfrac{1}{2}V'(x)\begin{bmatrix} 0 & 0 \\ 0 & x_1^2 \end{bmatrix} V'^{\mathrm{T}}(x) \\ + \frac{1}{2}\mathrm{tr}[g_1 x_1\ g_2 x_2]V''(x)\begin{bmatrix} g_1 x_1 \\ g_2 x_2 \end{bmatrix} + (2-g_1^2)x_1^2 + (2-g_2^2)x_2^2, \end{aligned} \tag{6.65}$$

which implies that $V'(x) = [2x_1,\ 2x_2]$. Furthermore, since $V(0) = 0$, $V(x) = x_1^2 + x_2^2$. Hence, the optimal feedback control law is given by $\phi(x) = -\frac{1}{2}R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) = -2x_1 x_2$.

Finally, note that (6.48) implies

$$\begin{aligned} \mathcal{L}V(x) &= \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix}\begin{bmatrix} -x_1 + x_1 x_2^2 \\ -x_2 - 2x_1^2 x_2 \end{bmatrix} + g_1^2 x_1^2 + g_2^2 x_2^2 \\ &= -(2-g_1^2)x_1^2 - (2-g_2^2)x_2^2 - 2x_1^2 x_2^2 < 0, \end{aligned} \tag{6.66}$$

for all $(x_1, x_2) \neq (0, 0)$, and hence, $\phi(x_1, x_2) = -2x_1x_2$ is a global stabilizer for (6.62) and (6.63). Now, with $L_1(x) > 0$ and $L_2(x) = 0$, (6.56) is always satisfied with $\theta \in (0, 1)$. Therefore the largest value that θ can attain such that (6.56) holds is $\theta_{\max} = 1$, which leads to a disk margin of $(\frac{1}{2}, \infty)$. $\triangle$

Next, we consider the case in which $R_2(x)$, $x \in \mathbb{R}^n$, is not a diagonal constant matrix. For the following result define

$$\bar{\gamma} \triangleq \sup_{x \in \mathbb{R}^n} \sigma_{\max}(R_2(x)), \qquad \underline{\gamma} \triangleq \inf_{x \in \mathbb{R}^n} \sigma_{\min}(R_2(x)), \tag{6.67}$$

where $R_2(x)$ is such that $\bar{\gamma} < \infty$ and $\underline{\gamma} > 0$.

Theorem 6.6. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (6.11) and (6.12), where $\phi(x)$ is a stochastically stabilizing feedback control law given by (6.33) and where $V(x)$, $x \in \mathbb{R}^n$, satisfies (6.34). If there exists $\theta \in \mathbb{R}$ such that $0 < \theta < 1$ and (6.56) is satisfied, then the nonlinear stochastic system $\mathcal{G}$ has a disk margin $(\frac{1}{1+\eta\theta}, \frac{1}{1-\eta\theta})$, where $\eta \triangleq \sqrt{\underline{\gamma}/\bar{\gamma}}$.

Proof. Note that for all $u(\cdot) \in \mathcal{U}$, it follows from Lemma 6.1 that the solution $x(t)$, $t \geq 0$, to (6.11) satisfies

$$\begin{aligned} \mathcal{L}V(x) &\leq [u + y]^{\mathrm{T}} R_2(x)[u + y] - \theta^2 u^{\mathrm{T}} R_2(x) u \\ &\leq \bar{\gamma}[u + y]^{\mathrm{T}}[u + y] - \underline{\gamma}\theta^2 u^{\mathrm{T}} u. \end{aligned} \tag{6.68}$$

Hence, with the storage function $V_{\mathrm{s}}(x) = \frac{1}{2\underline{\gamma}} V(x)$, it follows from Proposition 6.1 that $\mathcal{G}$ is stochastically dissipative with respect to the supply rate $r(u, y) = u^{\mathrm{T}} y + \frac{1-\eta^2\theta^2}{2} u^{\mathrm{T}} u + y^{\mathrm{T}} y$. Now, the result is a direct consequence of Definition 6.5 with $\alpha = \frac{1}{1+\eta\theta}$ and $\beta = \frac{1}{1-\eta\theta}$. $\square$

Next, we provide an alternative result that guarantees sector and gain margins for the case in which $R_2(x)$, $x \in \mathbb{R}^n$, is diagonal.

Theorem 6.7. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (6.11) and (6.12), where $\phi(x)$ is a stochastically stabilizing feedback control law given by (6.33) and where $V(x)$, $x \in \mathbb{R}^n$, satisfies (6.34). Furthermore, let $R_2(x) = \mathrm{diag}\,[r_1(x), \ldots, r_m(x)]$, where $r_i : \mathbb{R}^n \to \mathbb{R}$, $r_i(x) > 0$, $i = 1, \ldots, m$. If $\mathcal{G}$ is zero-state observable and there exists $\theta \in \mathbb{R}$ such that $0 < \theta < 1$ and

$$(1-\theta^2)L_1(x) - \tfrac{1}{4}L_2(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) \geq 0, \qquad x \in \mathbb{R}^n, \tag{6.69}$$

then the nonlinear stochastic dynamical system $\mathcal{G}$ has a sector (and, hence, gain) margin $(\frac{1}{1+\theta}, \frac{1}{1-\theta})$.

Proof. Let $\Delta(-y) = \sigma(-y)$, where $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ is a static nonlinearity such that $\sigma(0) = 0$, $\sigma(v) = [\sigma_1(v_1), \ldots, \sigma_m(v_m)]^{\mathrm{T}}$, and $\alpha v_i^2 < \sigma_i(v_i)v_i < \beta v_i^2$, for all $v_i \neq 0$, $i = 1, \ldots, m$, where $\alpha = \frac{1}{1+\theta}$ and $\beta = \frac{1}{1-\theta}$; or, equivalently, $(\sigma_i(v_i) - \alpha v_i)(\sigma_i(v_i) - \beta v_i) < 0$, for all $v_i \neq 0$, $i = 1, \ldots, m$. In this case, the closed-loop system (6.11) and (6.12) with $u = \sigma(-y)$ is given by

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))\sigma(\phi(x(t)))]\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \stackrel{\text{a.s.}}{=} x_0, \quad t \geq 0. \tag{6.70}$$

Next, consider the Lyapunov function candidate $V(x)$, $x \in \mathbb{R}^n$, satisfying (6.34) and let $\mathcal{L}V(x)$ denote the Lyapunov infinitesimal generator of the closed-loop system (6.70). Now, it follows from (6.34) and (6.69) that

$$\begin{aligned}
\mathcal{L}V(x) &= V'(x)f(x) + V'(x)G(x)\sigma(\phi(x)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) \\
&\leq V'(x)f(x) + V'(x)G(x)\sigma(\phi(x)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) + L_1(x) \\
&\quad - \tfrac{1}{4(1-\theta^2)}L_2(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) \\
&\quad + (1-\theta^2)\left[\sigma(\phi(x)) + \tfrac{1}{2(1-\theta^2)}R_2^{-1}(x)L_2^{\mathrm{T}}(x)\right]^{\mathrm{T}} R_2(x) \\
&\quad \cdot \left[\sigma(\phi(x)) + \tfrac{1}{2(1-\theta^2)}R_2^{-1}(x)L_2^{\mathrm{T}}(x)\right] \\
&= V'(x)f(x) + L_1(x) + V'(x)G(x)\sigma(\phi(x)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x)
\end{aligned}$$

$$
\begin{aligned}
&\quad +(1-\theta^2)\sigma^{\mathrm{T}}(\phi(x))R_2(x)\sigma(\phi(x)) + L_2(x)\sigma(\phi(x)) \\
&= \phi^{\mathrm{T}}(x)R_2(x)\phi(x) - 2\phi^{\mathrm{T}}(x)R_2(x)\sigma(\phi(x)) \\
&\quad +(1-\theta^2)\sigma^{\mathrm{T}}(\phi(x))R_2(x)\sigma(\phi(x)) \\
&= \sum_{i=1}^{m} r_i(x)(\tfrac{1}{\beta}\sigma_i(-y_i)+y_i)(\tfrac{1}{\alpha}\sigma_i(-y_i)+y_i) \\
&= \tfrac{1}{\alpha\beta}\sum_{i=1}^{m} r_i(x)\left(\sigma_i(-y_i)+\alpha y_i\right)\left(\sigma_i(-y_i)+\beta y_i\right) \\
&\le 0, \quad x \in \mathbb{R}^n,
\end{aligned}
$$

which, by Theorem 3.1, implies that the closed-loop system (6.70) is Lyapunov stable in probability.

Next, it follows from [**?**, Cor. 4.1] that $\mathcal{L}V(x) \overset{\text{a.s.}}{\to} 0$ as $t \to \infty$, and note that $\mathcal{L}V(x) = 0$ if and only if $y = 0$. Now, since $\mathcal{G}$ is zero-state observable it follows that $x(t) \overset{\text{a.s.}}{\to} 0$ as $t \to \infty$. Thus, since $V_{\mathrm{s}}(\cdot)$ is radially unbounded, the closed-loop system (6.70) is globally asymptotically stable in probability for all $\sigma(\cdot)$ such that $\alpha v_i^2 < \sigma_i(v_i)v_i < \beta v_i^2$, $v_i \neq 0$, $i = 1, \ldots, m$, which implies that the nonlinear stochastic system $\mathcal{G}$ given by (6.11) and (6.12) has sector (and, hence, gain) margins (α, β). □

Note that in the case where $R_2(x)$, $x \in \mathbb{R}^n$, is diagonal, Theorem 6.7 guarantees larger gain and sector margins to the gain and sector margin guarantees provided by Theorem 6.6. However, Theorem 6.7 does not provide disk margin guarantees.

6.6. Inverse Optimality of Nonlinear Stochastic Feedback Regulators

In this section, we give sufficient conditions that guarantee that a given nonlinear feedback controller has prespecified disk, sector, and gain margins.

Proposition 6.2. Let $\theta \in (0, 1)$ and let $R_2 \in \mathbb{R}^{m\times m}$ be a positive-definite matrix. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (6.11) and (6.12), where $\phi(x)$ is a stochastically stabilizing feedback control law. Then there exist functions $V : \mathbb{R}^n \to \mathbb{R}$, $L_1 : \mathbb{R}^n \to \mathbb{R}$, and $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$ such that $\phi(x) = -\frac{1}{2}R_2^{-1}[V'(x)G(x) + L_2(x)]^{\mathrm{T}}$, $V(\cdot)$ is two-times continuously differentiable, $V(0) = 0$, $V(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, and, for all $x \in \mathbb{R}^n$,

$$
\begin{aligned}
0 &= V'(x)f(x) + L_1(x) - \tfrac{1}{4}[V'(x)G(x) + L_2(x)]R_2^{-1}[V'(x)G(x) + L_2(x)]^{\mathrm{T}} \\
&\quad + \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x), \qquad (6.71)\\
0 &\leq (1-\theta^2)L_1(x) - \tfrac{1}{4}L_2(x)R_2^{-1}L_2^{\mathrm{T}}(x), \qquad (6.72)
\end{aligned}
$$

if and only if, for all $u(\cdot) \in \mathcal{U}$, there exists $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$, $V(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, and the solution $x(t)$, $t \geq 0$, to (6.11) satisfies

$$\mathcal{L}V(x) \leq [u+y]^{\mathrm{T}}R_2(x)[u+y] - \theta^2 u^{\mathrm{T}}R_2(x)u. \qquad (6.73)$$

Proof. If there exist functions $V : \mathbb{R}^n \to \mathbb{R}$, $L_1 : \mathbb{R}^n \to \mathbb{R}$, and $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$ such that $\phi(x) = -\frac{1}{2}R_2^{-1}[V'(x)G(x) + L_2(x)]^{\mathrm{T}}$ and (6.71) and (6.72) are satisfied, then it follows from Lemma 6.1 that (6.73) is satisfied. Conversely, if for $u(\cdot) \in \mathcal{U}$ the solution $x(t)$, $t \geq 0$, to (6.11) satisfies (6.73), then with $Q = R_2$, $S = R_2$, and $R = (1-\theta^2)R_2$, it follows from (6.10) of Theorem 6.1 that

$$
\begin{aligned}
0 \geq{}& V'(x)f(x) + \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) - \phi^{\mathrm{T}}(x)R_2\phi(x) + \tfrac{1}{4(1-\theta^2)}[2\phi^{\mathrm{T}}(x)R_2 + V'(x)G(x)] \\
&\cdot R_2^{-1}[2\phi^{\mathrm{T}}(x)R_2 + V'(x)G(x)]^{\mathrm{T}}, \quad x \in \mathbb{R}^n.
\end{aligned}
$$

The result now follows with $L_1(x) = -V'(x)f(x) + \phi^{\mathrm{T}}(x)R_2\phi(x) - \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x)$ and $L_2(x) = -[2\phi^{\mathrm{T}}(x)R_2 + V'(x)G(x)]$. □

Note that if (6.71) and (6.72) are satisfied, then it follows from Theorem 6.3 that the feedback control law $\phi(x) = -\frac{1}{2}R_2^{-1}[V'(x)G(x)+L_2(x)]^{\mathrm{T}}$ minimizes the cost functional (6.27). Hence, Proposition 6.2 provides necessary and sufficient conditions for optimality of a given stochastically stabilizing feedback control law with prespecified disk margin guarantees.

The following result presents specific disk margin guarantees for inverse optimal controllers.

Theorem 6.8. Let $\theta \in (0,1)$ be given. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (6.11) and (6.12), where $\phi(x)$ is a stochastically stabilizing feedback control law. Assume that there exist functions $V : \mathbb{R}^n \to \mathbb{R}$ and $R_2 : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ such that $V(\cdot)$ is two-times continuously differentiable, $R_2(x) > 0$, $x \in \mathbb{R}^n$, and

$$V(0) = 0, \tag{6.74}$$

$$V(x) > 0, \qquad x \in \mathbb{R}^n, \qquad x \neq 0, \tag{6.75}$$

$$V'(x)[f(x) + G(x)\phi(x)] + \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) < 0, \qquad x \in \mathbb{R}^n, \qquad x \neq 0, \tag{6.76}$$

$$\begin{aligned} V'(x)f(x) &+ \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) - \phi^{\mathrm{T}}(x)R_2^{-1}(x)\phi(x) + \tfrac{1}{1-\theta^2}\Big(\phi^{\mathrm{T}}(x) + \tfrac{1}{2}V'(x)G(x) \\ &\cdot R_2^{-1}(x)\Big)R_2(x)\Big(\phi^{\mathrm{T}}(x) + \tfrac{1}{2}V'(x)G(x)R_2^{-1}(x)\Big)^{\mathrm{T}} \leq 0, \qquad x \in \mathbb{R}^n, \end{aligned} \tag{6.77}$$

and

$$V(x) \to \infty \text{ as } \|x\| \to \infty. \tag{6.78}$$

Then the nonlinear stochastic dynamical system $\mathcal{G}$ has a disk margin $(\frac{1}{1+\eta\theta}, \frac{1}{1-\eta\theta})$, where $\eta = \sqrt{\underline{\gamma}/\bar{\gamma}}$ and $\underline{\gamma}$ and $\bar{\gamma}$ are given by (6.67). Furthermore, with the feedback control law $\phi(x)$ the performance functional

$$\begin{aligned} J(x_0, u(\cdot)) \;=\; \mathbb{E}^{x_0}\bigg[\int_0^\infty &[-V'(x(t))(f(x(t)) + G(x(t))u(t)) \\ &+(\phi(x(t)) - u(t))^{\mathrm{T}}R_2(x(t))(\phi(x(t)) - u(t))]\mathrm{d}t\bigg] \end{aligned} \tag{6.79}$$

is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \qquad x_0 \in \mathbb{R}^n. \tag{6.80}$$

Proof. The result is a direct consequence of Theorems 6.3 and 6.6 with $L_1(x) = -V'(x)f(x)+\phi^{\mathrm{T}}(x)R_2(x)\phi(x)-\frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x)$ and $L_2(x) = -(2\phi^{\mathrm{T}}(x)R_2(x)+V'(x)G(x))$. Specifically, in this case, all the conditions of Theorem 6.3 are trivially satisfied. Furthermore, note that (6.77) is equivalent to (6.56). The result is now immediate. □

The next result provides sufficient conditions that guarantee that a given nonlinear feedback controller has prespecified gain and sector margins.

Theorem 6.9. Let $\theta \in (0,1)$ be given. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (6.11) and (6.12), where $\phi(x)$ is a stochastically stabilizing feedback control law. Assume there exist functions $R_2(x) = \mathrm{diag}[r_1(x), \ldots, r_m(x)]$, where $r_i : \mathbb{R}^n \to \mathbb{R}$, $r_i(x) > 0$, $i = 1, \ldots, m$, and $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(\cdot)$ is two-times continuously differentiable and satisfies (6.74)–(6.78). Then the nonlinear stochastic dynamical system $\mathcal{G}$ has a disk margin $(\frac{1}{1+\theta}, \frac{1}{1-\theta})$. Furthermore, with the feedback control law $\phi(x)$ the performance functional (6.79) is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{S}(x_0)} J(x_0, u(\cdot)), \qquad x_0 \in \mathbb{R}^n. \tag{6.81}$$

Proof. The result is a direct consequence of Theorems 6.3 and 6.7 with the proof being identical to the proof of Theorem 6.8. □

6.7. Linear-Quadratic Optimal Stochastic Regulators

In this section, we specialize Theorems 6.5 and 6.6 to the case of linear stochastic systems with multiplicative disturbance noise. Specifically, consider the stabilizable stochastic system given by

$$\mathrm{d}x(t) = [Ax(t) + Bu(t)]\mathrm{d}t + xg^{\mathrm{T}}\mathrm{d}w(t), \quad x(0) \overset{\mathrm{a.s.}}{=} x_0, \quad t \geq 0, \tag{6.82}$$

$$y(t) = -Kx(t), \tag{6.83}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $K \in \mathbb{R}^{m \times n}$, and $g \in \mathbb{R}^d$, and assume that (A, K) is detectable and the system (6.82) and (6.83) is asymptotically stable in probability with the feedback $u = -y$ or, equivalently, $\tilde{A} + BK$ is Hurwitz, where $\tilde{A} = A + \frac{1}{2}\|g\|^2 I_n$. Furthermore, assume that K is an optimal regulator that minimizes the quadratic performance functional given by

$$J(x_0, u(\cdot)) = \mathbb{E}^{x_0}\left[\int_0^\infty [x^{\mathrm{T}}(t)R_1x(t) + 2x^{\mathrm{T}}(t)R_{12}u(t) + u^{\mathrm{T}}(t)R_2u(t)]\mathrm{d}t\right], \tag{6.84}$$

where $R_1 \in \mathbb{R}^{n \times n}$, $R_{12} \in \mathbb{R}^{n \times m}$, and $R_2 \in \mathbb{R}^{m \times m}$ are such that $R_2 > 0$, $R_1 - R_{12}R_2^{-1}R_{12}^{\mathrm{T}} \geq 0$, and (A, R_1) is observable. In this case, it follows from Theorem 6.3 with $f(x) = Ax$,

$G(x) = B$, $L_1(x) = x^{\mathrm{T}} R_1 x$, $L_2(x) = 2x^{\mathrm{T}} R_{12}$, $R_2(x) = R_2$, $\phi(x) = Kx$, and $V(x) = x^{\mathrm{T}} P x$ that the optimal control law K is given by $K = -R_2^{-1}(B^{\mathrm{T}} P + R_{12})$, where $P > 0$ is the solution to the algebraic regulator Riccati equation given by

$$0 = (\tilde{A} - BR_2^{-1}R_{12}^{\mathrm{T}})^{\mathrm{T}} P + P(\tilde{A} - BR_2^{-1}R_{12}^{\mathrm{T}}) + R_1 - R_{12}R_2^{-1}R_{12}^{\mathrm{T}} - PBR_2^{-1}B^{\mathrm{T}} P. \tag{6.85}$$

The following results provide guarantees of disk, sector, and gain margins for the system (6.82) and (6.83).

Corollary 6.1. Consider the stochastic dynamical system with multiplicative noise given by (6.82) and (6.83) and with performance functional (6.84), and let

$$\sigma_{\max}^2(R_{12}) < \sigma_{\min}(R_1)\sigma_{\min}(R_2).$$

Then, with $K = -R_2^{-1}(B^{\mathrm{T}} P + R_{12})$, where $P > 0$ satisfies (6.85), the system (6.82) and (6.83) has disk margin (and, hence, sector and gain margins) $(\frac{1}{1+\eta\theta}, \frac{1}{1-\eta\theta})$, where

$$\eta = \frac{\sigma_{\min}(R_2)}{\sigma_{\max}(R_2)}, \qquad \theta = \left(1 - \frac{\sigma_{\max}^2(R_{12})}{\sigma_{\min}(R_1)\sigma_{\min}(R_2)}\right)^{1/2}. \tag{6.86}$$

Proof. The result is a direct consequence of Theorem 6.6 with $f(x) = Ax$, $G(x) = B$, $\phi(x) = Kx$, $V(x) = x^{\mathrm{T}} P x$, $L_1(x) = x^{\mathrm{T}} R_1 x$, and $L_2(x) = 2x^{\mathrm{T}} R_{12}$. Specifically, note that (6.85) is equivalent to (6.34). Now, with θ given by (6.86), it follows that $(1 - \theta^2)R_1 - R_{12}R_2^{-1}R_{12}^{\mathrm{T}} \geq 0$, and hence, (6.69) is satisfied so that all the conditions of Theorem 6.6 are satisfied. $\square$

Corollary 6.2. Consider the stochastic dynamical system with multiplicative noise given by (6.82) and (6.83) and with performance functional (6.84), and let $\sigma_{\max}^2(R_{12}) < \sigma_{\min}(R_1) \cdot \sigma_{\min}(R_2)$, where R_2 is diagonal. Then, with $K = -R_2^{-1}(B^{\mathrm{T}} P + R_{12})$, where $P > 0$ satisfies (6.85), the system (6.82) and (6.83) has structured disk margin (and, hence, gain and sector) margin $(\frac{1}{1+\theta}, \frac{1}{1-\theta})$, where

$$\theta = \left(1 - \frac{\sigma_{\max}^2(R_{12})}{\sigma_{\min}(R_1)\sigma_{\min}(R_2)}\right)^{1/2}. \tag{6.87}$$

Proof. The result is a direct consequence of Theorem 6.5 with $f(x) = Ax$, $G(x) = B$, $\phi(x) = Kx$, $V(x) = x^{\mathrm{T}}Px$, $L_1(x) = x^{\mathrm{T}}R_1x$, and $L_2(x) = 2x^{\mathrm{T}}R_{12}$. Specifically, note that (6.85) is equivalent to (6.34). Now, with θ given by (6.87), it follows that $(1-\theta^2)R_1 - R_{12}R_2^{-1}R_{12}^{\mathrm{T}} \geq 0$, and hence, (6.69) is satisfied so that all the conditions of Theorem 6.5 are satisfied. □

The gain margins obtained in Corollary 6.2 are precisely the gain margins given in [24] for deterministic linear-quadratic optimal regulators with cross-weighting terms in the performance criterion. Furthermore, since Corollary 6.2 guarantees structured disk margins of $(\frac{1}{1+\theta}, \frac{1}{1-\theta})$, it follows that the system has a phase margin ϕ given by

$$\cos(\phi) = 1 - \frac{\theta^2}{2}, \tag{6.88}$$

or, equivalently,

$$\sin\left(\frac{\phi}{2}\right) = \frac{\theta}{2}. \tag{6.89}$$

In the case where $R_{12} = 0$ it follows from (6.87) that $\theta = 1$, and hence, Corollary 6.2 guarantees a phase margin of $\pm 60^\circ$ in each input-output channel. In addition, requiring that $R_1 \geq 0$, it follows from Corollary 6.2 that the system given by (6.82) and (6.83) has a gain and sector margin of $(\frac{1}{2}, \infty)$.

6.8. Stability Margins, Meaningful Inverse Optimality, and Stochastic Dissipativity

In this section, we specialize the results of Section 6.3 to the case where $L(x, u)$ is nonnegative for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$. In the terminology of [40,126] this corresponds to a *meaningful cost functional*. Here, we assume $L_2(x) \equiv 0$ and $L_1(x) \geq 0$, $x \in \mathbb{R}^n$. In this case, we establish connections between stochastic dissipativity and optimality for nonlinear stochastic controllers. The first result specializes Theorem 6.3 to the case in which $L_2(x) \equiv 0$.

Theorem 6.10. Consider the nonlinear stochastic dynamical system (6.11) with performance functional (6.27) with $L_2(x) \equiv 0$ and $L_1(x) \geq 0$, $x \in \mathbb{R}^n$. Assume there exists a two-times continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$V(0) = 0, \tag{6.90}$$

$$V(x) > 0, \qquad x \in \mathbb{R}^n, \qquad x \neq 0, \tag{6.91}$$

$$\begin{aligned} 0 = {} & L_1(x) + V'(x)f(x) + \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) \\ & -\tfrac{1}{4}V'(x)G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{6.92}$$

and

$$V(x) \to \infty \text{ as } \|x\| \to \infty. \tag{6.93}$$

Furthermore, assume that the system (6.27) and (6.12) is zero-state observable with $y = L_1(x)$. Then the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the closed-loop system

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))\phi(x(t))]\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0. \tag{6.94}$$

is globally asymptotically stable in probability with the feedback control law

$$\phi(x) = -\tfrac{1}{2}R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x), \tag{6.95}$$

and the performance functional (6.27) is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{S}(x_0)} J(x_0, u(\cdot)), \qquad x_0 \in \mathbb{R}^n. \tag{6.96}$$

Finally,

$$J(x_0, \phi(x(\cdot))) = V(x_0), \qquad x_0 \in \mathbb{R}^n. \tag{6.97}$$

Proof. The proof is similar to the proof of Theorem 6.3. □

Next, we show that for a given nonlinear stochastic dynamical system $\mathcal{G}$ given by (6.11) and (6.12), there exists an equivalence between optimality and stochastic dissipativity. For the following result we assume that for a given nonlinear stochastic system (6.11), if there exists a feedback control law $\phi(x)$ that minimizes the performance functional (6.27) with $R_2(x) \equiv I$, $L_2(x) \equiv 0$, and $L_1(x) \geq 0$, $x \in \mathbb{R}^n$, then there exists a two-times continuously differentiable positive-definite function $V(x)$, $x \in \mathbb{R}^n$, such that (6.92) is satisfied.

Theorem 6.11. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (6.11) and (6.12). The feedback control law $u = \phi(x)$ is optimal with respect to a performance functional (6.26) with $R_2(x) \equiv I$, $L_2(x) \equiv 0$, and $L_1(x) \geq 0$, $x \in \mathbb{R}^n$, if and only if the nonlinear stochastic system $\mathcal{G}$ is stochastically dissipative with respect to the supply rate $r(u, y) = y^{\mathrm{T}}y + 2u^{\mathrm{T}}y$ and has a two-times continuously differentiable positive-definite, radially unbounded storage function $V(x)$, $x \in \mathbb{R}^n$.

Proof. If the control law $\phi(x)$ is optimal with respect to a performance functional (6.26) with $R_2(x) \equiv I$, $L_2(x) \equiv 0$, and $L_1(x) \geq 0$, $x \in \mathbb{R}^n$, then, by assumption, there exists a two-times continuously differentiable positive-definite function $V(x)$ such that (6.92) is satisfied. Hence, it follows from Proposition 6.2 that the solution $x(t)$, $t \geq 0$, to (6.11) satisfies

$$\mathcal{L}V(x) \leq [u + y]^{\mathrm{T}} R_2(x)[u + y] - \theta^2 u^{\mathrm{T}} R_2(x) u, \tag{6.98}$$

which implies, by Proposition 6.1, that $\mathcal{G}$ is stochastically dissipative with respect to the supply rate $r(u, y) = y^{\mathrm{T}}y + 2u^{\mathrm{T}}y$.

Conversely, if $\mathcal{G}$ is stochastically dissipative with respect to the supply rate $r(u, y) = y^{\mathrm{T}}y + 2u^{\mathrm{T}}y$ and has a two-times continuously differentiable positive-definite storage function, then, with $h(x) = -\phi(x)$, $J(x) \equiv 0$, $Q = I$, $R = 0$, and $S = 2I$, it follows from Theorem 6.1 that there exists a function $\ell : \mathbb{R}^n \to \mathbb{R}^p$ such that $\phi(x) = -\frac{1}{2}G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)$ and, for all $x \in \mathbb{R}^n$,

$$0 = V'(x)f(x) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) - \tfrac{1}{4}V'(x)G(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) + \ell^{\mathrm{T}}(x)\ell(x).$$

Now, the result follows from Theorem 6.10 with $L_1(x) = \ell^{\mathrm{T}}(x)\ell(x)$. □

Example 6.2. Consider the nonlinear stochastic dynamical system given by

$$\mathrm{d}x(t) = -x(t) + u(t) + gx(t)\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{6.99}$$

where $g < \sqrt{3}$, with performance functional

$$J(x_0, u(\cdot)) = \mathbb{E}^{x_0}\left[\int_0^\infty [(3 - g^2)x^2(t) + u^2(t)]\mathrm{d}t\right]. \tag{6.100}$$

To design an optimal control law $\phi(x)$ that minimizes (6.100) we use Theorem 6.4 with $f(x) = -x$, $G(x) = 1$, $D(x) = gx$, $L_1(x) = (3 - g^2)x^2$, $L_2(x) = 0$, and $R_2(x) = 1$. Now, note that (6.49) holds with $V(x) = x^2$. Therefore, the optimal control law is given by

$$\phi(x) = -\tfrac{1}{2}R_2^{-1}(x)[V'(x)G(x) + L_2(x)]^{\mathrm{T}} = -x. \tag{6.101}$$

Now, from Proposition 6.2, since (6.71) and (6.72) hold with $\theta = 1$, we have

$$\mathcal{L}V(x) \leq (u + y)^2 - u^2 = y^2 + 2uy, \tag{6.102}$$

where $y = -\phi(x) = x$, which implies, by Proposition 6.1, that $\mathcal{G}$ is stochastically dissipative with respect to the supply rate $r(u, y) = y^2 + 2uy$. △

The next result gives disk and structured disk margins for the nonlinear stochastic dynamical system $\mathcal{G}$ given by (6.11) and (6.12).

Corollary 6.3. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (6.11) and (6.27), where $\phi(x)$ is a stochastically stabilizing feedback control law given by (6.33) with $L_2(x) \equiv 0$ and where $V(x)$, $x \in \mathbb{R}^n$, satisfies (6.34). Furthermore, assume $R_2(x) = \mathrm{diag}[r_1, \ldots, r_m]$, where $r_i > 0$, $i = 1, \ldots, m$, and $L_1(x) \geq 0$, $x \in \mathbb{R}^n$. Then the nonlinear stochastic dynamical system $\mathcal{G}$ has a structured disk margin $(\frac{1}{2}, \infty)$. If, in addition, $R_2(x) \equiv I_m$, then the nonlinear stochastic system $\mathcal{G}$ has a disk margin $(\frac{1}{2}, \infty)$

Proof. The result is a direct consequence of Theorem 6.5. Specifically, if $L_1(x) \geq 0$, $x \in \mathbb{R}^n$, and $L_2(x) \equiv 0$, then (6.56) is trivially satisfied for all $\theta \in (0, 1)$. Now, the result follows immediately by letting $\theta \to 1$. □

Finally, we provide sector and gain margins for the nonlinear stochastic dynamical system $\mathcal{G}$ given by (6.11) and (6.12).

Corollary 6.4. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (6.11) and (6.12), where $\phi(x)$ is a stochastically stabilizing feedback control law given by (6.33) with

$L_2(x) \equiv 0$ and where $V(x)$, $x \in \mathbb{R}^n$, satisfies (6.34). Furthermore, assume $R_2(x) = \mathrm{diag}[r_1(x), \ldots, r_m(x)]$, where $r_i : \mathbb{R}^n \to \mathbb{R}$, $r_i(x) > 0$, $i = 1, \ldots, m$, and $L_1(x) \geq 0$, $x \in \mathbb{R}^n$. Then the nonlinear stochastic dynamical system $\mathcal{G}$ has a sector (and, hence, gain) margin $(\frac{1}{2}, \infty)$.

Proof. The result is a direct consequence of Theorem 6.7. Specifically, if $L_1(x) \geq 0$, $x \in \mathbb{R}^n$, and $L_2(x) \equiv 0$, then (6.56) is trivially satisfied for all $\theta \in (0, 1)$. Now, the result follows immediately by letting $\theta \to 1$. $\square$

Chapter 7

Universal Feedback Controllers and Inverse Optimality for Nonlinear Stochastic Systems

7.1. Introduction

The consideration of Lyapunov functions for proving stability of feedback dynamical systems is one of the cornerstones of systems and control theory. For dynamical systems with continuously differentiable flows, the concept of smooth control Lyapunov functions was developed by Artstein [6] to show the existence of a feedback stabilizing controller. A constructive feedback control law based on a universal construction of smooth control Lyapunov functions was given by Sontag [128]. An extended notion of nonsmooth control Lyapunov functions as well as a universal feedback controller for discontinuous dynamical systems based on the existence of nonsmooth Lyapunov functions defined in the sense of generalized Clarke gradients and set-valued Lie derivatives was developed in [49, 120–122].

The aforementioned results on control Lyapunov functions along with the constructive feedback control laws predicated on these generalized energy functions are developed for deterministic dynamical systems. In numerous applications where dynamical models are used to describe the behavior of natural and engineering systems, stochastic components and random disturbances are often incorporated into the models. The stochastic aspects of the models are used to quantify system uncertainty as well as the dynamic relationships of sequences of random events between system-environment interactions. In [21, 36, 37] the authors provide Lyapunov-like techniques for stochastic stabilization. Specifically, asymptotic stability in probability of affine in the control stochastic dynamical systems using stochastic control Lyapunov functions leading to the existence of smooth, except possibly at the equilibrium point of the system, stochastically stabilizing feedback control laws are provided.

In this chapter, we build on the results of [21, 36, 37] as well as on the recent stochastic finite time stabilization framework of [116] to develop a constructive universal feedback control law for stochastic finite time stabilization of stochastic dynamical systems. In addition, we present necessary and sufficient conditions for continuity of such controllers. Finally, we show that for every nonlinear stochastic dynamical system for which a stochastic control Lyapunov function can be constructed there exists an inverse optimal feedback control law in the sense of [53, 115] with guaranteed sector and gain margins of $(\frac{1}{2}, \infty)$.

7.2. Notation, Definitions, and Mathematical Preliminaries

For the results in the chapter involving finite time stability we assume that the uniform Lipschitz continuity condition (3.3) and the growth condition (3.4) are satisfied for all x, $y \in \mathcal{D}\backslash\{0\}$. Furthermore, we assume that for every initial condition $x_0 \in \mathcal{D}\backslash\{0\}$, (3.1) has a unique solution in forward time. Analogous assumptions are made for the controlled problem.

Definition 7.1 [116]. The zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to (3.1) is (*globally*) *stochastically finite-time stable* if there exists an operator $T : \mathcal{H}_n \to \mathcal{H}_1^{[0,\infty)}$, called the *stochastic settling-time operator*, such that the following statements hold.

i) *Finite-time convergence in probability.* For every $x(0) \in \mathcal{H}_n$, $s^{x(0)}(t)$ is defined on $[0, T(x(0)))$, $s^{x(0)}(t) \in \mathcal{H}_n$ for all $t \in [0, T(x(0)))$, and

$$\mathbb{P}^{x_0}\left(\lim_{t \to T(x(0))} \|s^{x(0)}(t)\| = 0\right) = 1.$$

ii) *Lyapunov stability in probability.* For every $\varepsilon > 0$,

$$\mathbb{P}^{x_0}\left(\sup_{t\in[0,T(x(0)))} \|s^{x(0)}(t)\| > \varepsilon\right) = 0.$$

Equivalently, for every $\varepsilon > 0$ and $\rho \in (0,1)$, there exist $\delta = \delta(\varepsilon,\rho) > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(0)$, $\mathbb{P}^{x_0}\left(\sup_{t\in[0,T(x(0)))} \|s^{x(0)}(t)\| > \varepsilon\right) \leq \rho$.

iii) *Finiteness of the stochastic settling-time operator.* For every $x \in \mathcal{H}_n$ the stochastic settling-time operator $T(x)$ exists and is finite with probability one, that is, $\mathbb{E}^x\left[T(x)\right] < \infty$.

It is easy to see from Definition 7.1 that

$$T(x(0)) = \inf\{t \in \overline{\mathbb{R}}_+ : s(t,x(0)) = 0\}, \quad x(0) \in \mathcal{H}_n^{\mathbb{R}^n}.$$

Proposition 7.1. Suppose the origin is a stochastically finite time stable equilibrium of (3.1) and let $T : \mathcal{H}_n \to \mathcal{H}_1^{[0,\infty]}$ be the stochastic finite time operator. Then the following statements hold.

i) If $\tau \geq 0$ and $x(0) \in \mathcal{H}_n$, then $T(s(\tau,x(0))) \overset{\text{a.s.}}{=} \max\{T(x(0)) - \tau, 0\}$.

ii) $T(\cdot)$ is sample continuous on $\mathcal{H}_n$ if and only if $T(\cdot)$ is sample continuous at 0.

Proof. The proof is a direct consequence of Proposition 3.2 given in [116] and, hence, is omitted. □

Next, we present a sufficient condition for global stochastic finite time stability.

Theorem 7.1. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (3.1) with $\mathcal{D} = \mathbb{R}^n$. If there exist a radially unbounded positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ and a function $\eta : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ such that $V(0) = 0$, $V(x)$ is two-times continuously differentiable for all $x \in \mathbb{R}^n$, $\eta(\cdot)$ is continuously differentiable, and, for all $x \in \mathbb{R}^n$,

$$V'(x)f(x) + \frac{1}{2}\text{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) \leq -\eta(V(x)), \tag{7.1}$$

$$\int_0^{\varepsilon} \frac{\mathrm{d}v}{\eta(v)} < \infty, \quad \varepsilon \in [0, \infty), \tag{7.2}$$

$$\eta'(v) > 0, \quad v \geq 0, \tag{7.3}$$

then $\mathcal{G}$ is globally stochastically finite time stable. Moreover, there exists a settling-time operator $T : \mathcal{H}_n \to \mathcal{H}_1^{[0,\infty)}$ such that

$$\mathbb{E}^{x_0}[T(x_0)] \leq \int_0^{V(x_0)} \frac{\mathrm{d}v}{\eta(v)}, \quad x_0 \in \mathbb{R}^n. \tag{7.4}$$

Proof. The proof follows as a special case of the proof of Theorem 3.1 of [116]. □

Remark 7.1. If $\eta(V) = cV^{\theta}$, where $c > 0$ and $\theta \in (0, 1)$, then $\eta(\cdot)$ satisfies (7.2) and (7.3). In this case, (7.4) becomes

$$\mathbb{E}^{x_0}[T(x(0))] \leq \frac{V(x_0)^{1-\theta}}{c(1-\theta)}.$$

For deterministic dynamical systems, this specialization recovers the finite time stability results given in [12].

Finally, we consider the controlled nonlinear stochastic dynamical system given by

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))u(t)]\,\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{7.5}$$

$$y(t) = -\phi(x(t)), \tag{7.6}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}^m$, with a nonlinear-nonquadratic performance criterion

$$J(x_0, u(\cdot)) = \mathbb{E}^{x_0}\left[\int_0^{\infty} [L_1(x(t)) + u^{\mathrm{T}}(t)R_2(x(t))u(t)]\mathrm{d}t\right], \tag{7.7}$$

where $L_1 : \mathbb{R}^n \to \mathbb{R}$ and $R_2 : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ are such that $L_1(x) \geq 0$, $x \in \mathbb{R}^n$, and $R_2(x) > 0$, $x \in \mathbb{R}^n$. In this case, the optimal nonlinear feedback controller $u = \phi(x)$ that minimizes the nonlinear-nonquadratic performance criterion (7.7) is given by the following result. For the statement of this result recall the definition of the set of stochastic regulation controllers given by

$$\mathcal{S}(x_0) \triangleq \Big\{u(\cdot) : u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (7.5) is such that } \mathbb{P}^{x_0}\Big(\mathfrak{B}_{x_0}^{u(\cdot)}\Big) = 1,$$

$$x_0 \in \mathbb{R}^n, \text{ where } \mathfrak{B}_{x_0}^{u(\cdot)} \triangleq \left\{ x(\{t \geq t_0\}, \omega) : \lim_{t\to\infty} \|x(t,\omega)\| = 0, \, \omega \in \Omega \right\} \Bigg\}.$$

Theorem 7.2. Consider the nonlinear stochastic dynamical system (7.5) with performance functional (7.7) with $L_1(x) \geq 0$, $x \in \mathbb{R}^n$. Assume there exists a two-times continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$V(0) = 0, \tag{7.8}$$

$$V(x) > 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{7.9}$$

$$\begin{aligned} 0 = {} & L_1(x) + V'(x)f(x) + \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) \\ & -\tfrac{1}{4}V'(x)G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{7.10}$$

and

$$V(x) \to \infty \text{ as } \|x\| \to \infty. \tag{7.11}$$

Furthermore, assume that the system (7.7) and (7.6) is zero-state observable with $y = L_1(x)$. Then the zero solution $x(t) \stackrel{\text{a.s.}}{\equiv} 0$ of the closed-loop system

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))\phi(x(t))]\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \stackrel{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{7.12}$$

is globally asymptotically stable in probability with the feedback control law

$$\phi(x) = -\tfrac{1}{2}R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x), \tag{7.13}$$

and the performance functional (7.6) is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \quad x_0 \in \mathbb{R}^n. \tag{7.14}$$

Finally,

$$J(x_0, \phi(x(\cdot))) = V(x_0), \quad x_0 \in \mathbb{R}^n. \tag{7.15}$$

Proof. The proof is similar to the proof of Theorem 8.3 for the deterministic optimal control problem given in [51]. □

Finally, we provide sector and gain margins for the nonlinear stochastic dynamical system $\mathcal{G}$ given by (7.5) and (7.6).

Theorem 7.3. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (7.5) and (7.6) where $\phi(x)$ is a stabilizing feedback control law given by (7.13) and where $V(x)$, $x \in \mathbb{R}^n$, satisfies (7.10). Furthermore, assume $R_2(x) = \mathrm{diag}[r_1(x), \ldots, r_m(x)]$, where $r_i : \mathbb{R}^n \to \mathbb{R}$, $r_i(x) > 0$, $i = 1, \ldots, m$, and $L_1(x) \geq 0$, $x \in \mathbb{R}^n$. Then the nonlinear dynamical system $\mathcal{G}$ has a sector (and, hence, gain) margin $(\frac{1}{2}, \infty)$.

Proof. The result is a direct consequence of Theorem 6.4 of [53]. □

7.3. Stochastic Control Lyapunov Functions

In this section, we consider a feedback control problem and introduce the notion of *stochastic control Lyapunov functions*. Furthermore, using the concept of stochastic control Lyapunov functions we provide necessary and sufficient conditions for stochastic nonlinear system stabilization.

Consider the nonlinear stochastic controlled dynamical system $\mathcal{G}$ given by

$$\mathrm{d}x(t) = F(x(t), u(t))\mathrm{d}t + D(x(t), u(t))\mathrm{d}w(t), \quad x(t_0) \overset{\text{a.s.}}{=} x_0, \quad t \geq t_0, \tag{7.16}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in \mathcal{H}_m^U$, $U \subseteq \mathbb{R}^m$, $F : \mathcal{D} \times U \to \mathbb{R}^n$, and $D : \mathcal{D} \times U \to \mathbb{R}^{n \times d}$. Here we assume that $u(\cdot)$ satisfies sufficient regularity conditions such that (7.16) has a unique solution forward in time. Specifically, we assume that the control process $u(\cdot)$ in (7.16) is restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ adapted to the filtration $\{\mathcal{F}_t\}_{t \geq t_0}$ such that $u(t) \in \mathcal{H}_m$, $t \geq t_0$, and, for all $t \geq s$, $w(t) - w(s)$ is independent of $u(\tau)$, $w(\tau)$, $\tau \leq s$, and $x(t_0)$, and hence, $u(\cdot)$ is nonanticipative. Furthermore, we assume that $u(\cdot)$ takes values in a compact, metrizable set $\mathcal{U}$ and the uniform Lipschitz continuity and growth conditions (3.3) and (3.4) hold for the controlled drift and diffusion terms $F(x, u)$ and $D(x, u)$ uniformly in u. In this case, it follows from Theorem 2.2.4 of [4] that there exists a pathwise unique solution to (7.16) in $(\Omega, \{\mathcal{F}_{t \geq t_0}\}, \mathbb{P}^{x_0})$.

A measurable function $\phi : \mathcal{D} \to U$ satisfying $\phi(0) = 0$ is called a *control law*. If $u(t) = \phi(x(t))$, $t \geq t_0$, where $\phi(\cdot)$ is a control law and $x(t)$, $t \geq t_0$, satisfies (7.16), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot)$ has values in U. Given a control law $\phi(\cdot)$ and a feedback control law $u(t) = \phi(x(t))$, $t \geq t_0$, the *closed-loop system* (7.16) has the form

$$\mathrm{d}x(t) = F(x(t), \phi(x(t)) + D(x(t), \phi(x(t))\mathrm{d}w(t), \quad x(t_0) \stackrel{\text{a.s.}}{=} x_0, \quad t \geq t_0. \tag{7.17}$$

The following two definitions are required for stating the results of this section.

Definition 7.2. Let $\phi : \mathcal{D} \to U$ be a measurable mapping on $\mathcal{D}\backslash\{0\}$ with $\phi(0) = 0$. Then (7.16) is *stochastically feedback asymptotically stabilizable* if the zero solution $x(t) \stackrel{\text{a.s.}}{\equiv} 0$ of the closed-loop system (7.17) is stochastically asymptotically stable.

Definition 7.3 [37]. Consider the controlled nonlinear stochastic dynamical system given by (7.16). A two-times continuously differentiable positive-definite function $V : \mathcal{D} \to \mathbb{R}$ satisfying

$$\inf_{u\in U}\left[V'(x)F(x,u) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x,u)V''(x)D(x,u)\right] < 0, \quad x \in \mathcal{D}, \quad x \neq 0, \tag{7.18}$$

is called a *stochastic control Lyapunov function.*

Note that if (7.18) holds, then there exists a feedback control law $\phi : \mathcal{D} \to U$ such that $V'(x)F(x,\phi(x)) + \frac{1}{2}\mathrm{tr}D^{\mathrm{T}}(x,\phi(x))V''(x)D(x,\phi(x)) < 0$, $x \in \mathcal{D}$, $x \neq 0$, and hence, Theorem **??** implies that if there exists a stochastic control Lyapunov function for the nonlinear stochastic dynamical system (7.16), then there exists a feedback control law $\phi(x)$ such that the zero solution $x(t) \stackrel{\text{a.s.}}{\equiv} 0$ of the closed-loop nonlinear stochastic dynamical system (7.16) is stochastically asymptotically stable. Conversely, if there exists a feedback control law $u = \phi(x)$ such that the zero solution $x(t) \stackrel{\text{a.s.}}{\equiv} 0$ of the nonlinear stochastic dynamical system (7.16) is stochastically asymptotically stable and $D(x)$, $x \in \mathbb{R}^n$, satisfies a nondegeneracy condition, then it follows from Theorem 3.2 of [54, pp. 165] that there exists a two-times continuously differentiable positive-definite function $V : \mathcal{D} \to \mathbb{R}$ such that $V'(x)F(x, \phi(x)) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x, \phi(x))V''(x)D(x, \phi(x)) < 0$, $x \in \mathcal{D}$, $x \neq 0$, or, equivalently, there exists a stochastic control Lyapunov function for the nonlinear stochastic dynamical system (7.16). Hence, a given nonlinear stochastic dynamical system of the form (7.16) is stochastically feedback asymptotically stabilizable if and only if there exists a stochastic control Lyapunov function satisfying (7.18). Finally, in the case where $\mathcal{D} = \mathbb{R}^n$ and $U = \mathbb{R}^m$ the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to (7.16) is globally stochastically asymptotically stabilizable if and only if $V(x) \to \infty$ as $\|x\| \to \infty$.

Next, we consider the special case of nonlinear stochastic affine systems in the control and construct state feedback controllers that globally stochastically asymptotically stabilize the zero solution of the nonlinear stochastic dynamical system under the assumption that the system has a radially unbounded stochastic control Lyapunov function. Specifically, we consider nonlinear stochastic affine systems of the form

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))u(t)]\,\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{7.19}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $f(0) = 0$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $D : \mathbb{R}^n \to \mathbb{R}^d$, and $f(\cdot)$, $G(\cdot)$, and $D(\cdot)$ are continuous functions.

Theorem 7.4. Consider the controlled nonlinear stochastic dynamical system given by (7.19). Then a two-times continuously differentiable positive-definite, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ is a stochastic control Lyapunov function of (7.19) if and only if

$$V'(x)f(x) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) < 0, \quad x \in \mathcal{R}, \tag{7.20}$$

where $\mathcal{R} \triangleq \{x \in \mathbb{R}^n,\ x \neq 0 : \quad V'(x)G(x) = 0\}$.

Proof. The proof is a direct consequence of the definition of a stochastic control Lyapunov function by noting that for systems of the form (7.19),

$$\inf_{u\in\mathbb{R}^m} \Big[V'(x)[f(x) + G(x)u] + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x)\Big] = -\infty, \quad x \notin \mathcal{R}, \quad x \neq 0.$$

Hence, (7.18) is equivalent to (7.20), which proves the result. $\square$

It follows from Theorem 7.4 that the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of a nonlinear stochastic affine system of the form (7.19) is globally stochastically feedback asymptotically stabilizable if and only if there exists a two-times continuously differentiable positive-definite, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying (7.20). Hence, Theorem 7.4 provides necessary and sufficient conditions for nonlinear stochastic system stabilization.

Next, using Theorem 7.4 we *construct* an explicit feedback control law that is a function of the stochastic control Lyapunov function $V(\cdot)$. Specifically, consider the feedback control law given by

$$\phi(x) = \begin{cases} -\left(c_0 + \frac{(\alpha(x)+\xi(x))+\sqrt{(\alpha(x)+\xi(x))^2+(\beta^{\mathrm{T}}(x)\beta(x))^2}}{\beta^{\mathrm{T}}(x)\beta(x)}\right)\beta(x), & \beta(x) \neq 0, \\ 0, & \beta(x) = 0, \end{cases} \tag{7.21}$$

where $\alpha(x) \triangleq V'(x)f(x)$, $\beta(x) \triangleq G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)$, $\xi(x) \triangleq \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x)$, and $c_0 \geq 0$. In this case, the stochastic control Lyapunov function $V(\cdot)$ of (7.19) is a Lyapunov function for the closed-loop system (7.19) with $u = \phi(x)$, where $\phi(x)$ is given by (7.21). In particular, the infinitesimal generator $\mathcal{L}V(\cdot)$ of the nonlinear stochastic dynamical system (7.19) with $u = \phi(x)$ given by (7.21) is given by

$$\begin{aligned} \mathcal{L}V(x) &\triangleq V'(x)[f(x) + G(x)\phi(x)] + \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) \\ &= \alpha(x) + \beta^{\mathrm{T}}(x)\phi(x) + \xi(x) \\ &= \begin{cases} -c_0\beta^{\mathrm{T}}(x)\beta(x) - \sqrt{(\alpha(x)+\xi(x))^2 + (\beta^{\mathrm{T}}(x)\beta(x))^2}, & \beta(x) \neq 0, \\ \alpha(x) + \xi(x), & \beta(x) = 0, \end{cases} \\ &< 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \end{aligned} \tag{7.22}$$

which implies that $V(\cdot)$ is a Lyapunov function for the closed-loop system (7.19) guaranteeing global stochastic asymptotic stability with $u = \phi(x)$ given by (7.21).

Since $f(\cdot)$, $G(\cdot)$, and $D(\cdot)$ are smooth it follows that $\alpha(x)$, $\beta(x)$, and $\xi(x)$, $x \in \mathbb{R}^n$, are smooth functions, and hence, $\phi(x)$ given by (7.21) is smooth for all $x \in \mathbb{R}^n$ if either $\beta(x) \neq 0$ or $\alpha(x) + \xi(x) < 0$. Hence, the feedback control law given by (7.21) is smooth everywhere except for the origin. The following result provides necessary and sufficient conditions under which the feedback control law given by (7.21) is guaranteed to be continuous and Lipschitz continuous at the origin in addition to being smooth everywhere else.

Theorem 7.5. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (7.19) with a radially unbounded stochastic control Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$. Then the following statements hold.

i) The control law $\phi(x)$ given by (7.21) is continuous at $x = 0$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $0 < \|x\| < \delta$, there exists $u \in \mathbb{R}^m$ such that $\|u\| < \varepsilon$ and $\alpha(x) + \beta^{\mathrm{T}}(x)u + \xi(x) < 0$.

ii) There exists a stabilizing control law $\hat{\phi}(x)$ such that $\alpha(x) + \beta^{\mathrm{T}}(x)\hat{\phi}(x) + \xi(x) < 0$, $x \in \mathbb{R}^n$, $x \neq 0$, and $\hat{\phi}(x)$ is Lipschitz continuous at $x = 0$ if and only if the control law $\phi(x)$ given by (7.21) is Lipschitz continuous at $x = 0$.

Proof. Necessity of *i*) is trivial with $u = \phi(x)$. Conversely, assume that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $0 < \|x\| < \delta$, there exists $u \in \mathbb{R}^m$ such that $\|u\| < \varepsilon$ and $\alpha(x) + \beta^{\mathrm{T}}(x)u + \xi(x) < 0$. In this case, since $\|u\| < \varepsilon$ it follows from the Cauchy-Schwarz inequality that $\alpha(x) + \xi(x) < \varepsilon\|\beta(x)\|$. Furthermore, since $V(\cdot)$ is two-times continuously differentiable and $G(\cdot)$ is continuous it follows that there exists $\hat{\delta} > 0$ such that for all $0 < \|x\| < \hat{\delta}$, $\|\beta(x)\| < \varepsilon$. Hence, for all $0 < \|x\| < \delta_{\min}$, where $\delta_{\min} \triangleq \min\{\delta, \hat{\delta}\}$, it follows that $\alpha(x) + \xi(x) < \varepsilon\|\beta(x)\|$ and $\|\beta(x)\| < \varepsilon$.

Furthermore, if $\beta(x) = 0$, then $\|\phi(x)\| = 0$, and if $\beta(x) \neq 0$, then it follows from (7.21) that

$$\begin{aligned}\|\phi(x)\| &\leq c_0\|\beta(x)\| + \frac{|\alpha(x) + \xi(x) + \sqrt{(\alpha(x) + \xi(x))^2 + (\beta^{\mathrm{T}}(x)\beta(x))^2}|}{\|\beta(x)\|} \\ &\leq \frac{2(\alpha(x) + \xi(x)) + (c_0 + 1)\|\beta(x)\|^2}{\|\beta(x)\|} \\ &\leq (c_0 + 3)\varepsilon, \quad 0 < \|x\| < \delta_{\min}, \quad \alpha(x) + \xi(x) > 0,\end{aligned}$$

and

$$\begin{aligned}\|\phi(x)\| &\leq c_0\|\beta(x)\| + \frac{(\alpha(x)+\xi(x)) + \sqrt{(\alpha(x)+\xi(x))^2 + (\beta^{\mathrm{T}}(x)\beta(x))^2}}{\|\beta(x)\|} \\ &\leq c_0\|\beta(x)\| + \frac{\beta^{\mathrm{T}}(x)\beta(x)}{\|\beta(x)\|} \\ &= (c_0+1)\|\beta(x)\| < (c_0+1)\varepsilon, \quad 0 < \|x\| < \delta_{\min}, \quad \alpha(x)+\xi(x) \leq 0.\end{aligned}$$

Hence, it follows that for every $\hat{\varepsilon} \triangleq (c_0+3)\varepsilon > 0$, there exists $\delta_{\min} > 0$ such that for all $\|x\| < \delta_{\min}$, $\|\phi(x)\| < \hat{\varepsilon}$, which implies that $\phi(\cdot)$ is continuous at the origin.

Next, to show necessity of *ii*) assume that there exists a stabilizing control $\hat{\phi}(x)$ such that $\alpha(x)+\beta^{\mathrm{T}}(x)\hat{\phi}(x)+\xi(x) < 0$, $x \in \mathbb{R}^n$, $x \neq 0$, and $\hat{\phi}(x)$ is Lipschitz continuous at $x = 0$ with a Lipschitz constant $\hat{L}$; that is, there exists $\delta > 0$ such that for all $x \in \mathcal{B}_\delta(0)$, $\|\hat{\phi}(x)\| \leq \hat{L}\|x\|$. Now, since $V(\cdot)$ is continuous and $V'(0) = 0$, it follows that there exists $K > 0$ such that $\|\beta(x)\| \leq K\|x\|$, $x \in \mathcal{B}_\delta(0)$. Hence,

$$\begin{aligned}\|\phi(x)\| &\leq c_0\|\beta(x)\| + \frac{|(\alpha(x)+\xi(x)) + \sqrt{(\alpha(x)+\xi(x))^2 + (\beta^{\mathrm{T}}(x)\beta(x))^2}|}{\|\beta(x)\|} \\ &\leq \frac{2(\alpha(x)+\xi(x)) + (c_0+1)\|\beta(x)\|^2}{\|\beta(x)\|} \\ &\leq (2\hat{L} + (c_0+1)K)\|x\|, \quad x \in \mathcal{B}_\delta(0), \quad \alpha(x)+\xi(x) > 0,\end{aligned}$$

and

$$\begin{aligned}\|\phi(x)\| &\leq c_0\|\beta(x)\| + \frac{(\alpha(x)+\xi(x)) + \sqrt{(\alpha(x)+\xi(x))^2 + (\beta^{\mathrm{T}}(x)\beta(x))^2}}{\|\beta(x)\|} \\ &\leq c_0\|\beta(x)\| + \frac{\beta^{\mathrm{T}}(x)\beta(x)}{\|\beta(x)\|} \\ &= (c_0+1)\|\beta(x)\| \\ &< (c_0+1)K\|x\|, \quad x \in \mathcal{B}_\delta(0), \quad \alpha(x)+\xi(x) \leq 0,\end{aligned}$$

which implies that for all $x \in \mathcal{B}_\delta(0)$, $\|\phi(x)\| \leq L\|x\|$, where $L \triangleq 2\hat{L} + (c_0+1)K$, and hence, $\phi(\cdot)$ is Lipschitz continuous.

Finally, sufficiency of *ii*) follows immediately with $\hat{\phi}(x) = \phi(x)$. $\square$

Next, we present sufficient conditions for stochastic finite time stabilization using a control Lyapunov function involving a scalar differential inequality.

Theorem 7.6. Consider the nonlinear stochastic dynamical system (7.19). Assume there exists a two-times continuously differentiable function $V : \mathcal{D} \to \overline{\mathbb{R}}_+$ such that $V(\cdot)$ is positive definite and

$$V'(x)f(x) + \frac{1}{2}\mathrm{tr}D^{\mathrm{T}}(x)V''(x)D(x) \leq -c(V(x))^{\alpha}, \quad x \in \mathcal{R}, \tag{7.23}$$

where $c > 0$, $\alpha \in (0, 1)$, and $\mathcal{R} \triangleq \{x \in \mathbb{R}^n, x \neq 0 : V'(x)G(x) = 0\}$. Then the nonlinear stochastic dynamical system (7.19) with the feedback controller $u = \phi(x)$, $x \in \mathbb{R}^n$, given by

$$\phi(x) = \begin{cases} -\left(c_0 + \frac{(\alpha(x)+\xi(x)+c(V(x))^{\alpha}) + \sqrt{(\alpha(x)+\xi(x)+c(V(x))^{\alpha})^2+(\beta^{\mathrm{T}}(x)\beta(x))^2}}{\beta^{\mathrm{T}}(x)\beta(x)}\right)\beta(x), & \beta(x) \neq 0, \\ 0, & \beta(x) = 0, \end{cases} \tag{7.24}$$

where $c_0 > 0$, $\alpha(x) \triangleq V'(x)f(x)$, $x \in \mathbb{R}^n$, $\beta(x) \triangleq G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)$, $x \in \mathbb{R}^n$, and $\xi(x) \triangleq \frac{1}{2}D^{\mathrm{T}}(x)V''(x)D(x)$, $x \in \mathbb{R}^n$, is stochastically finite time stable and there exists a stochastic settling time operator $T : \mathcal{H}_m \to \mathcal{H}_1^{[0,\infty)}$ such that

$$\mathbb{E}^{x_0}[T(x_0)] \leq \frac{1}{c(1-\alpha)}(V(x_0))^{1-\alpha}, \quad x_0 \in \mathbb{R}^n. \tag{7.25}$$

Furthermore, $V(\cdot)$ is a stochastic control Lyapunov function.

Proof. The infinitesimal generator $\mathcal{L}(\cdot)$ of the closed-loop stochastic dynamical system (7.19), with $u = \phi(x)$, $x \in \mathbb{R}^n$, given by (7.24), is given by

$$\begin{aligned}
\mathcal{L}V(x) &= V'(x)f(x) + V'(x)G(x)\phi(x) + \frac{1}{2}D^{\mathrm{T}}(x)V''(x)D(x) \\
&= \alpha(x) + \beta^{\mathrm{T}}(x)\phi(x) + \xi(x) \\
&= \begin{cases} -c_0\beta^{\mathrm{T}}(x)\beta(x) - \sqrt{(\alpha(x)+\xi(x)+c(V(x))^{\alpha})^2 + (\beta^{\mathrm{T}}(x)\beta(x))^2} \\ \quad -c(V(x))^{\alpha}, & \beta(x) \neq 0, \\ \alpha(x) + \xi(x), & \beta(x) = 0, \end{cases} \\
&< -c(V(x))^{\alpha}, \quad x \in \mathbb{R}^n.
\end{aligned} \tag{7.26}$$

Now, it follows from Theorem 7.1 with $\eta(V) = cV^{\theta}$ that the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to (7.19) is stochastically finite time stable with the stochastic settling time $\mathbb{E}^{x_0}[T(x_0)] \leq \frac{1}{c(1-\alpha)}(V(x_0))^{1-\alpha}$, $x_0 \in \mathbb{R}^n$. In this case, it follows from Definition 7.3 that $V(x)$, $x \in \mathbb{R}^n$, is a stochatic control Lyapunov function. □

Since $f(\cdot)$, $G(\cdot)$, and $D(\cdot)$ are continuous and $V(\cdot)$ is two-times continuously differentiable, it follows that $\alpha(x)$, $\beta(x)$, and $\xi(x)$, $x \in \mathbb{R}^n$, are continuous functions, and hence, $\phi(x)$ given by (7.24) is continuous for all $x \in \mathbb{R}^n$ if either $\beta(x) \neq 0$ or $\alpha(x) + \xi(x) + c(V(x))^{\alpha} < 0$ for all $x \in \mathbb{R}^n$. Hence, the feedback control law given by (7.24) is continuous everywhere except for the origin. However, as shown in Theorem 7.5, the feedback control law $\phi(x)$ given in (7.24) is continuous on $\mathbb{R}^n$ if and only if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $0 < \|x\| < \delta$ there exists $u \in \mathbb{R}^m$ such that $\|u\| < \varepsilon$ and $\alpha(x) + \beta^{\mathrm{T}}(x)u + \xi(x) + c(V(x))^{\alpha} < 0$.

7.4. Meaningful Inverse Optimality and Control Lyapunov Functions

In this section, we show that given a stochastic control Lyapunov function for a controlled nonlinear stochastic dynamical system, the feedback control law given by (7.21) guarantees sector and gain margins of $(\frac{1}{2}, \infty)$.

Theorem 7.7. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (7.5) and let the two-times continuously differentiable positive-definite, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ be a stochastic control Lyapunov function of (7.5), that is,

$$V'(x)f(x) + \frac{1}{2}\mathrm{tr}D^{\mathrm{T}}(x)V''(x)D(x) < 0, \quad x \in \mathcal{R}, \tag{7.27}$$

where $\mathcal{R} \triangleq \{x \in \mathbb{R}^n : \ x \neq 0, \ V'(x)G(x) = 0\}$. Then with the feedback stabilizing control law given by

$$\phi(x) = \begin{cases} -\left(c_0 + \frac{(\alpha(x)+\xi(x))+\sqrt{(\alpha(x)+\xi(x))^2+(\beta^{\mathrm{T}}(x)\beta(x))^2}}{\beta^{\mathrm{T}}(x)\beta(x)}\right)\beta(x), & \beta(x) \neq 0, \\ 0, & \beta(x) = 0, \end{cases} \tag{7.28}$$

where $\alpha(x) \triangleq V'(x)f(x)$, $\beta(x) \triangleq G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)$, $\xi(x) = \frac{1}{2}\mathrm{tr}D^{\mathrm{T}}(x)V''(x)D(x)$, and $c_0 > 0$, the nonlinear stochastic dynamical system $\mathcal{G}$ given by (7.5) and (7.6) has a sector (and, hence, gain) margin $(\frac{1}{2}, \infty)$. Furthermore, with the feedback control law $u = \phi(x)$ the performance functional

$$J(x_0, u(\cdot)) = \mathbb{E}^{x_0}\left[\int_0^\infty [\alpha(x(t)) + \xi(x(t)) - \tfrac{\gamma(x(t))}{2}\beta^{\mathrm{T}}(x(t))\beta(x(t)) + \tfrac{1}{2\gamma(x(t))}u^{\mathrm{T}}(t)u(t)]\mathrm{d}t\right], \tag{7.29}$$

where

$$\gamma(x) \triangleq \begin{cases} \left(c_0 + \frac{(\alpha(x)+\xi(x))+\sqrt{(\alpha(x)+\xi(x))^2+(\beta^{\mathrm{T}}(x)\beta(x))^2}}{\beta^{\mathrm{T}}(x)\beta(x)}\right), & \beta(x) \neq 0, \\ c_0, & \beta(x) = 0, \end{cases} \tag{7.30}$$

is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u \in \mathcal{S}(x_0)} J(x_0, u(\cdot)), \quad x_0 \in \mathbb{R}^n. \tag{7.31}$$

Proof. The result is a direct consequence of Theorems 7.2 and 7.3 with $R_2(x) = \frac{1}{2\gamma(x)}I_m$ and $L_1(x) = -(\alpha(x)+\xi(x))+\frac{\gamma(x)}{2}\beta^{\mathrm{T}}(x)\beta(x)$. Specifically, it follows from (7.30) that $R_2(x) > 0$, $x \in \mathbb{R}^n$, and

$$\begin{aligned} L_1(x) &= -(\alpha(x) + \xi(x)) + \tfrac{\gamma(x)}{2}\beta^{\mathrm{T}}(x)\beta(x) \\ &= \begin{cases} \frac{1}{2}\left(c_0\beta^{\mathrm{T}}(x)\beta(x) - (\alpha(x) + \xi(x)) + \sqrt{(\alpha(x) + \xi(x))^2 + (\beta^{\mathrm{T}}(x)\beta(x))^2}\right), & \beta(x) \neq 0, \\ -(\alpha(x) + \xi(x)), & \beta(x) = 0. \end{cases} \end{aligned} \tag{7.32}$$

Now, it follows from (7.32) that $L_1(x) \geq 0$, $\beta(x) \neq 0$, and, since $V(\cdot)$ is a stochastic control Lyapunov function of (7.5), it follows from Theorem 7.4 that $L_1(x) = -(\alpha(x) + \xi(x)) \geq 0$ for all $x \in \mathcal{R} = \{x \in \mathbb{R}^n : \ x \neq 0, \ \beta(x) = 0\}$. Hence, (7.32) yields $L_1(x) \geq 0$, $x \in \mathbb{R}^n$, so that all conditions of Theorem 7.3 are satisfied. $\square$

Theorem 7.7 shows that given a nonlinear stochastic dynamical system for which a stochastic control Lyapunov function can be constructed, the feedback control law given by (7.28) is inverse optimal with respect to a meaningful cost functional and has a sector (and, hence, gain) margin $(\frac{1}{2}, \infty)$.

Remark 7.2. Using the stochastic finite time optimal feedback control framework developed in [116], the stochastic finite time controller (7.24) can also be shown to be inverse optimal with respect to a *meaningful* (in the terminology of [115]) nonlinear-nonquadratic perfromance functional with guaranteed sector and gain margins. However, due to the space limitations we do not present this result here.

7.5. Illustrative Numerical Example

Our example considers control of thermoacoustic instabilities in combustion processes. Engineering applications involving steam and gas turbines and jet and ramjet engines for power generation and propulsion technology involve combustion processes. Due to the inherent coupling between several intricate physical phenomena in these processes involving acoustics, thermodynamics, fluid mechanics, and chemical kinetics, the dynamic behavior of combustion systems is characterized by highly complex nonlinear models [7, 8, 28, 65]. The unstable dynamic coupling between heat release in combustion processes generated by reacting mixtures releasing chemical energy and unsteady motions in the combustor develop acoustic pressure and velocity oscillations, which can severely impact operating conditions and system performance. These pressure oscillations, known as *thermoacoustic instabilities*, often lead to high vibration levels causing mechanical failures, high levels of acoustic noise, high burn rates, and even component melting. Hence, the need for active control to mitigate combustion-induced pressure instabilities is critical.

In this section, we design a finite-time stabilizing controller for a two-mode, nonlinear time-averaged combustion model with nonlinearities present due to the second-order gas dynamics. This model is developed in [28] and is given by

$$\mathrm{d}x_1(t) = (\alpha_1 x_1(t) + \theta_1 x_2(t) - \beta(x_1(t)x_3(t) + x_2(t)x_4(t)) + u_1(t))\mathrm{d}t + \sigma_1 x_1(t)\mathrm{d}w(t),$$
$$x_1(0) \overset{\text{a.s.}}{=} x_{10}, \quad t \geq 0, \quad (7.33)$$
$$\mathrm{d}x_2(t) = (-\theta_1 x_1(t) + \alpha_1 x_2(t) + \beta(x_2(t)x_3(t) - x_1(t)x_4(t)) + u_2(t))\mathrm{d}t + \sigma_2 x_2(t)\mathrm{d}w(t),$$

$$x_2(0) \stackrel{\text{a.s.}}{=} x_{20}, \quad (7.34)$$

$$\mathrm{d}x_3(t) = (\alpha_2 x_3(t) + \theta_2 x_4(t) + \beta(x_1^2(t) - x_2^2(t)) + u_3(t))\mathrm{d}t + \sigma_3 x_3(t)\mathrm{d}w(t),$$

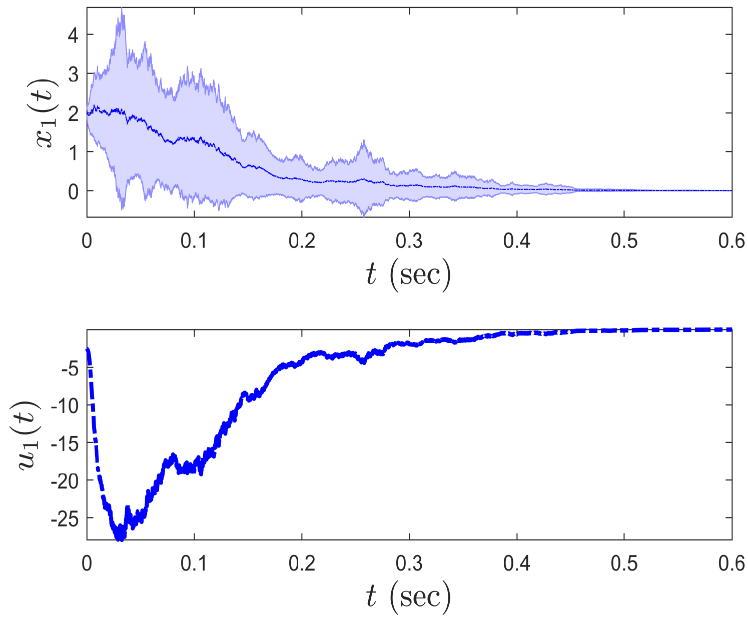
$$x_3(0) \stackrel{\text{a.s.}}{=} x_{30}, \quad (7.35)$$

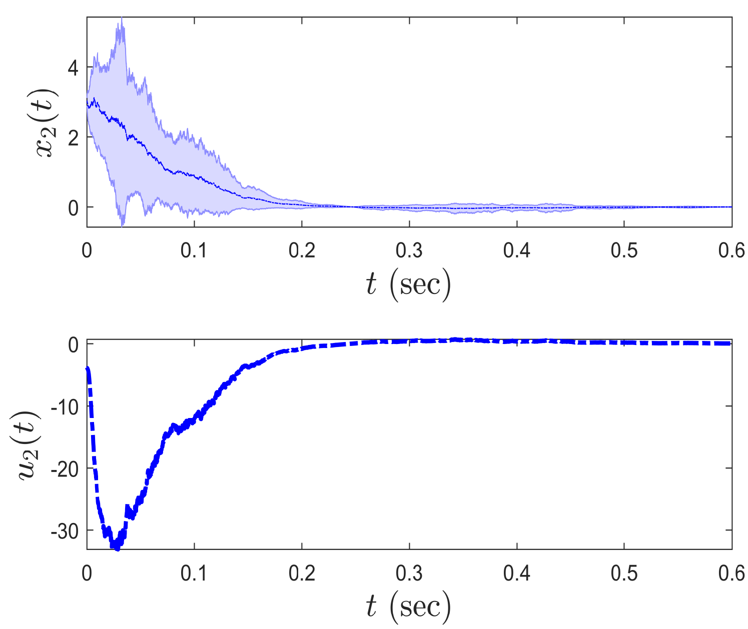
$$\mathrm{d}x_4(t) = (-\theta_2 x_3(t) + \alpha_2 x_4(t) + 2\beta x_1(t) x_2(t) + u_4(t))\mathrm{d}t + \sigma_4 x_4(t)\mathrm{d}w(t),$$

$$x_4(0) \stackrel{\text{a.s.}}{=} x_{40}, \quad (7.36)$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ represent growth/decay constants, $\theta_1, \theta_2 \in \mathbb{R}$ represent frequency shift constants, $\beta = ((\gamma+1)/8\gamma)\omega_1$, where γ denotes the ratio of specific heats, ω_1 is the frequency of the fundamental mode, σ_1, σ_2, σ_3, and $\sigma_4 \in \mathbb{R}$ represent augmentation factors of the variance of the state dependent stochastic disturbance, and u_i, $i = 1, \ldots, 4$, are control input signals. For the data parameters $\alpha_1 = 5$, $\alpha_2 = -55$, $\theta_1 = 4$, $\theta_2 = 32$, $\gamma = 1.4$, $\omega_1 = 1$, $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 1$, and $x_0 = [2, 3, 1, 1]^{\mathrm{T}}$, the open-loop (i.e., $u_i(t) \equiv 0, i = 1, \ldots, 4$) dynamics (7.33)–(7.36) result in sustained oscillations.

To stabilize this system in finite time we design a feedback control law given by (7.24), where $V(x) = \frac{1}{2}x^{\mathrm{T}}x$, $x \in \mathbb{R}^4$, $c = 1$, $c_0 = 1$, $\alpha = \frac{3}{4}$. In this case, $V'(x) = x^{\mathrm{T}}$, $G(x) = I_4$, and hence, $\mathcal{R} = \{x \in \mathbb{R}^4, x \neq 0 : x^{\mathrm{T}} = 0\} = \emptyset$. Thus, condition (7.23) is trivially satisfied and it follows from Theorem 7.6 that the closed-loop system (7.33)–(7.36) with the feedback control law (7.24) is finite time stable with $\mathbb{E}^{x_0}[T(x_0)] \leq 6.6195$. Figure 7.5.1 shows a sample trajectory along with the standard deviation of the state trajectories for $x_0 = [2, 3, 1, 1]^{\mathrm{T}}$ of the controlled system versus time along with the mean control signal versus time for 30 sample paths.


$x_1(t)$
t (sec)
$u_1(t)$
t (sec)


$x_2(t)$
t (sec)
$u_2(t)$
t (sec)

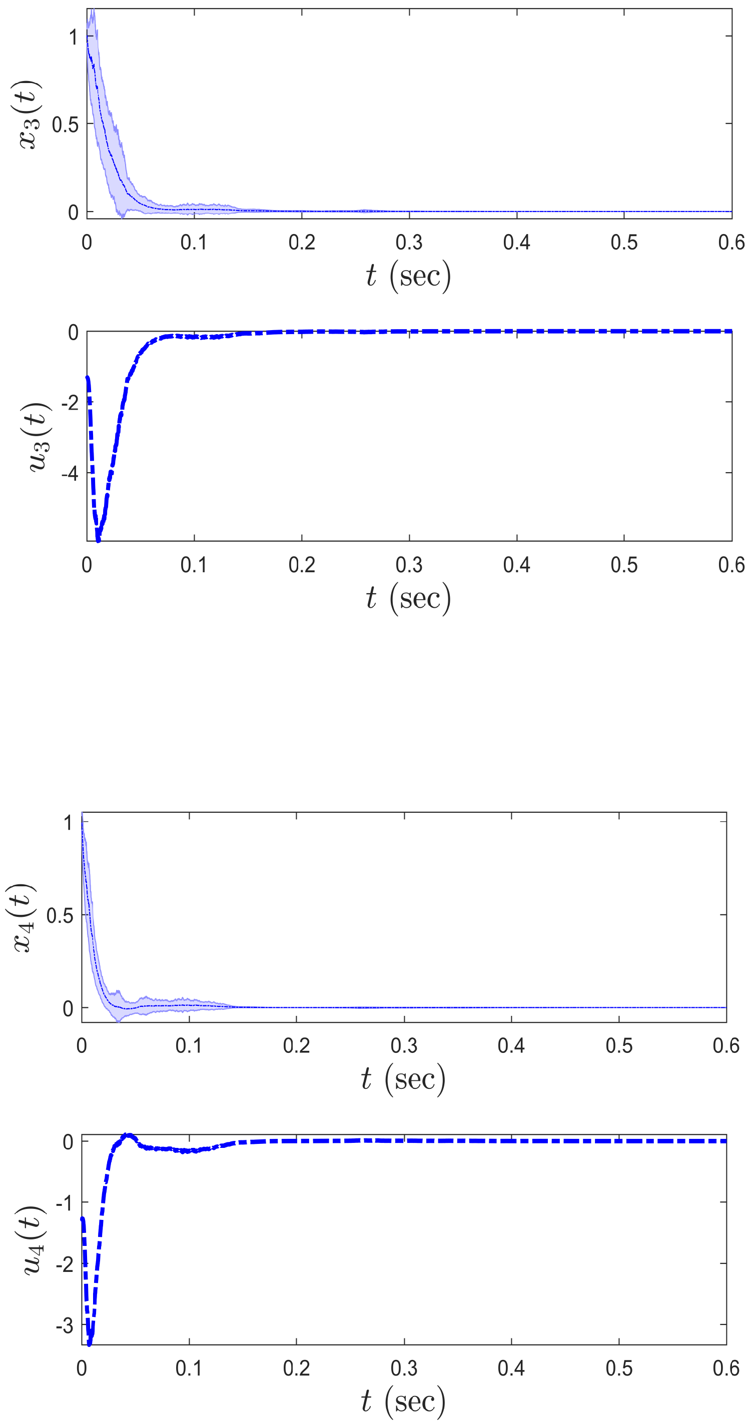


Figure 7.5.1: A sample trajectory along with the sample standard deviation of the closed-loop system trajectories versus time. The control profile is plotted as the mean of the 30 sample runs.

Chapter 8

Stochastic Semistability and Finite Time Semistability with Application to Consensus on with Communication Uncertainty

8.1. Introduction

For deterministic dynamical systems the authors in [13, 14, 51, 60] developed a unified stability analysis framework for systems having a continuum of equilibria. Since every neighborhood of a nonisolated equilibrium contains another equilibrium, a nonisolated equilibrium cannot be asymptotically stable nor finite time stable. Hence, asymptotic and finite time stability are not the appropriate notions of stability for systems having a continuum of equilibria. Two notions that are of particular relevance to such systems are convergence and semistability. Convergence is the property whereby every system solution converges (asymptotically or in finite time) to a limit point that may depend on the system initial condition. Semistability (resp., finite time semistability) is the additional requirement that all solutions converge asymptotically (resp., in finite time) to limit points that are Lyapunov stable. Semistability (resp., finite time semistability) for an equilibrium thus implies Lyapunov stability, and is implied by asymptotic (resp., finite time) stability.

It is important to note that semistability is not merely equivalent to asymptotic stability of the set of equilibria. Indeed, it is possible for a trajectory to converge to the set of equilibria without converging to any one equilibrium point [13]. Conversely, semistability does not imply that the equilibrium set is asymptotically stable in any accepted sense. This is because stability of sets is defined in terms of distance (especially in case of noncompact sets), and it is possible to construct examples in which the dynamical system is semistable, but the domain of semistability contains no ε-neighborhood (defined in terms of the distance) of the (noncompact) equilibrium set, thus ruling out asymptotic stability of the equilibrium set. Hence, semistability and set stability of the equilibrium set are independent notions.

In this chapter, we extend the theories of semistability and finite-time semistability for deterministic dynamical systems developed in [13, 14, 51, 60] to develop a rigorous framework for stochastic semistability and stochastic finite-time semistability. First, in Section 8.2, we extend the theory of stochastic semistability given in [112] by presenting new Lyapunov theorems as well as the first converse Lyapunov theorem for stochastic semistability, which holds with a continuous Lyapunov function whose infinitesimal generator decreases along the stochastic dynamical system trajectories and is such that the Lyapunov function satisfies inequalities involving the average distance to the set of equilibria. It is important to note here that stochastic semistability theory as developed in [112] involves a stronger set of stability in probability definitions that do not allow for a small probability of escape of the system sample trajectories for small deviations from the system equilibrium. While our stochastic semistability results developed in this paper resemble the results in [112], the proofs of our results are rendered more difficult by the fact that the results in this chapter are predicated on a weaker set of stability in probability definitions, and hence, provide a stronger set of stochastic semistability results.

Next, in Section 8.3, we establish stochastic finite time semistability theory. In particular, we present the notions of finite time convergence in probability and finite time semistability in probability for nonlinear stochastic dynamical systems driven by Markov diffusion processes. Furthermore, we establish the continuity of a settling time operator and develop a sufficient Lyapunov stability theorem for finite time semistability in probability. Specifically, we develop almost sure finite time convergence and stochastic Lyapunov stability properties to address almost sure finite time semistability requiring that the sample trajectories of a nonlinear stochastic dynamical system converge almost surely in finite time to a set of equilibrium solutions, wherein every equilibrium solution in the set is almost surely Lyapunov stable.

Next, in Sections 8.4 and 8.5, we use the results of Sections 8.2 and 8.3 to develop a general, thermodynamically motivated framework for designing semistable and finite-time semistable protocols for stochastic dynamical networks for achieving coordination tasks asymptotically and in finite time. Network systems involve distributed decision-making for coordination of networks of dynamic agents and address a broad area of applications including cooperative control of unmanned air vehicles (UAV's) and autonomous underwater vehicles (AUV's) for combat, surveillance, and reconnaissance [150], distributed reconfigurable sensor networks for managing power levels of wireless networks [25], air and ground transportation systems for air traffic control and payload transport and traffic management [134], swarms of air and space vehicle formations for command and control between heterogeneous air and space vehicles [35,140], and congestion control in communication networks for routing the flow of information through a network [103].

Even though convergence, semistability, finite time semistability, and optimality for deterministic multiagent network systems involving cooperative control tasks such as formation control, rendezvous, flocking, cyclic pursuit, and consensus have received considerable attention in the literature (see, for example, [1, 19, 20, 25, 32, 35, 44, 58–60, 64, 68, 76, 78, 79, 81, 88, 95, 96, 103, 118, 119, 127, 133–136, 139, 140, 150, 157], stochastic multiagent networks have not been as plethorically developed; notable contributions include [83, 84, 151, 152, 161]. These contributions address asymptotic convergence [161], time-varying network topologies [84], communication delays [152], asynchronous switchings [151], and optimality [83]; however, none of the aforementioned references address the problems of stochastic semistability and stochastic finite time semistability.

A unique feature of the closed-loop dynamics under any control algorithm that achieves consensus in a dynamical network is the existence of a continuum of equilibria representing a state of consensus. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial system state as well. Thus, from a practical viewpoint, it is not sufficient for a nonlinear control protocol to only guarantee that a network converges to a state of consensus since steady state convergence is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from a state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable, and consequently, semistable.

To capture network system uncertainty and communication uncertainty between the agents in a network, wherein the evolution of each link of the network communication topology follows a Markov process for modeling unknown communication noise and attenuations, we use the results of Sections 8.2 and 8.3 to develop almost sure consensus protocols for multiagent systems with nonlinear stochastic dynamics. Specifically, we use our stochastic semistability and stochastic finite time semistability frameworks to design distributed asymptotic and finite time consensus control protocols for nonlinear bidirectional dynamical networks with stochastic communication uncertainty. The proposed controller architectures are predicated on the recently developed notion of stochastic dynamical thermodynamics [50,113] resulting in controller architectures involving the exchange of generalized charge or energy state information between agents that guarantee that the closed-loop dynamical network is consistent with basic thermodynamic principles.

8.2. Stochastic Semistability

In this section, we develop a stability analysis framework for stochastic systems having a continuum of equilibria. Specifically, we present necessary and sufficient conditions for *stochastic semistability.* To develop stochastic semistability theory, we need some additional notation and definitions.

The measurable map $s : [0, \tau_x) \times \mathcal{D} \times \Omega \to \mathcal{D}$ denotes the *dynamic* or *flow* of the stochastic dynamical system (3.1) and, for all $t, \tau \in [0, \tau_x)$, satisfies the *cocycle* property $s(\tau, s(t, x), \omega) = s(t + \tau, x, \omega)$ and the *identity* (on $\mathcal{D}$) property $s(0, x, \omega) = x$ for all $x \in \mathcal{D}$ and $\omega \in \Omega$. The measurable map $s_t \triangleq s(t, \cdot, \omega) : \mathcal{D} \to \mathcal{D}$ is continuously differentiable for all $t \in [0, \tau_x)$ outside a $\mathbb{P}$-nullset and the sample path trajectory $s^x \triangleq s(\cdot, x, \omega) : [0, \tau_x) \to \mathcal{D}$ is continuous in $\mathcal{D}$ for all $t \in [0, \tau_x)$. Thus, for every $x \in \mathcal{D}$, there exists a trajectory of measures defined for all $t \in [0, \tau_x)$ satisfying the dynamical processes (3.1) with initial condition $x(0) \overset{\text{a.s.}}{=} x_0$. For simplicity of exposition we write $s(t, x)$ for $s(t, x, \omega)$ omitting its dependence on ω.

Next, the following definitions for limit sets and stochastic invariance are needed.

Definition 8.1. A point $p \in \mathcal{D}$ is a *limit point* of the trajectory $s(\cdot, x)$ of (3.1) if there exists a monotonic sequence $\{t_n\}_{n=0}^{\infty}$ of positive numbers, with $t_n \to \infty$ as $n \to \infty$, such that $s(t_n, x) \overset{\text{a.s.}}{\to} p$ as $n \to \infty$. The set of all limit points of $s(t, x), t \geq 0$, is the *limit set* $\omega(x)$ of $s(\cdot, x)$ of (3.1).

Definition 8.2 [87]. An open set $\mathcal{D} \subset \mathbb{R}^n$ is said to be *positively invariant with respect to* (3.1) if $\mathcal{D}$ is Borel and, for all $x_0 \in \mathcal{D}$, $\mathbb{P}^{x_0}(x(t) \in \mathcal{D}) = 1,\ t \geq t_0$.

It is important to note that the ω-limit set of a stochastic dynamical system is a ω-limit set of a trajectory of measures, that is, $p \in \omega(x)$ is a weak limit of a sequence of measures taken along every sample continuous bounded trajectory of (3.1). It can be shown that the ω-limit set of a stationary stochastic dynamical system attracts bounded sets and is measurable with respect to the σ-algebra of invariant sets. Thus, the measures of the stochastic process $x(\cdot)$ tend to an invariant set of measures and $x(t)$ asymptotically tends to the closure of the support set of this set of measures almost surely.

However, unlike deterministic dynamical systems, wherein ω-limit sets serve as global attractors, in stochastic dynamical systems stochastic invariance (see Definition 8.2) leads to ω-limit sets being defined for each fixed sample $\omega \in \Omega$ of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and hence, are pathwise attractors. This is due to the fact that a cocycle property rather than a semigroup property holds for stochastic dynamical systems. For details, see [18, 26, 27].

The following proposition gives a sufficient condition for a trajectory of (3.1) to converge almost surely to a limit point. For this result, $\mathcal{D}_{\rm c} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$ denotes a positively invariant set with respect to (3.1) and $s_t(\mathcal{H}_n^{\mathcal{D}_{\rm c}})$ denotes the image of $\mathcal{H}_n^{\mathcal{D}_{\rm c}} \subset \mathcal{H}_n^{\mathcal{D}}$ under the flow $s_t : \mathcal{H}_n^{\mathcal{D}_{\rm c}} \to \mathcal{H}_n^{\mathcal{D}}$; that is, $s_t(\mathcal{H}_n^{\mathcal{D}_{\rm c}}) \triangleq \{y : y = s_t(x_0) \text{ for some } x(0) \overset{\rm a.s.}{=} x_0 \in \mathcal{H}_n^{\mathcal{D}_{\rm c}}\}$.

Proposition 8.1. Consider the nonlinear stochastic dynamical system (3.1) and let $x \in \mathcal{D}_{\rm c}$. If the limit set $\omega(x)$ of (3.1) contains a Lyapunov stable in probability equilibrium point y, then $\lim_{x\to y} \mathbb{P}^x\big(\|\lim_{t\to\infty} s(t,x) - y\| = 0\big) = 1$, that is, $\omega(x) \overset{\rm a.s.}{=} \{y\}$ as $x \to y$.

Proof. Suppose $y \in \omega(x)$ is Lyapunov stable in probability and let $\mathcal{N}_\varepsilon \subseteq \mathcal{D}_{\rm c}$ be an open neighborhood of y. Since y is Lyapunov stable in probability, there exists an open neighborhood $\mathcal{N}_\delta \subset \mathcal{D}_{\rm c}$ of y such that $s_t(\mathcal{H}_n^{\mathcal{N}_\delta}) \subseteq \mathcal{H}_n^{\mathcal{N}_\varepsilon}$ as $x \to y$ for every $t \geq 0$. Now, since $y \in \omega(x)$, it follows that there exists $\tau \geq 0$ such that $s(\tau, x) \in \mathcal{H}_n^{\mathcal{N}_\delta}$. Hence, $s(t+\tau, x) = s_t(s(\tau,x)) \in s_t(\mathcal{H}_n^{\mathcal{N}_\delta}) \subseteq \mathcal{H}_n^{\mathcal{N}_\varepsilon}$ for every $t > 0$. Since $\mathcal{N}_\varepsilon \subseteq \mathcal{D}_{\rm c}$ is arbitrary, it follows that $y \overset{\rm a.s.}{=} \lim_{t\to\infty} s(t,x)$. Thus, $\lim_{n\to\infty} s(t_n, x) \overset{\rm a.s.}{=} y$ as $x \to y$ for every sequence $\{t_n\}_{n=1}^\infty$, and hence, $\omega(x) \overset{\rm a.s.}{=} \{y\}$ as $x \to y$. $\square$

The following definition introduces the notion of stochastic semistability.

Definition 8.3. An equilibrium solution $x(t) \overset{\rm a.s.}{\equiv} x_{\rm e} \in \mathcal{E}$ of (3.1) is *stochastically semistable* if the following statements hold.

i) For every $\varepsilon > 0$, $\lim_{x_0\to x_{\rm e}} \mathbb{P}^{x_0}\left(\sup_{0\leq t<\infty} \|x(t) - x_{\rm e}\| > \varepsilon\right) = 0$. Equivalently, for every

$\varepsilon > 0$ and $\rho \in (0,1)$, there exists $\delta = \delta(\varepsilon, \rho) > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(x_e)$,

$$\mathbb{P}^{x_0}\left(\sup_{0\le t<\infty} \|x(t) - x_e\| > \varepsilon\right) \le \rho.$$

ii) $\lim_{\mathrm{dist}(x_0,\mathcal{E})\to 0} \mathbb{P}^{x_0}\left(\lim_{t\to\infty} \mathrm{dist}(x(t),\mathcal{E}) = 0\right) = 1$. Equivalently, for every $\rho \in (0,1)$, there exists $\delta = \delta(\rho) > 0$ such that if $\mathrm{dist}(x_0,\mathcal{E}) \le \delta$, then $\mathbb{P}^{x_0}\left(\lim_{t\to\infty} \mathrm{dist}(x(t),\mathcal{E}) = 0\right) \ge 1-\rho$. The dynamical system (3.1) is *stochastically semistable* if every equilibrium solution of (3.1) is stochastically semistable. Finally, the dynamical system (3.1) is *globally stochastically semistable* if *i*) holds and $\mathbb{P}^{x_0}\left(\lim_{t\to\infty} \mathrm{dist}(x(t),\mathcal{E}) = 0\right) = 1$ for all $x_0 \in \mathbb{R}^n$.

Note that if $x(t) \overset{\text{a.s.}}{\equiv} x_e \in \mathcal{E}$ only satisfies *i*) in Definition 8.3, then the equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_e \in \mathcal{E}$ of (3.1) is Lyapunov stable in probability.

Next, we present sufficient conditions for stochastic semistability.

Theorem 8.1. Consider the nonlinear stochastic dynamical system (3.1). Let $\mathcal{Q} \subseteq \mathbb{R}^n$ be an open neighborhood of $\mathcal{E}$ and assume that there exists a two-times continuously differentiable function $V : \mathcal{Q} \to \overline{\mathbb{R}}_+$ such that

$$V'(x)f(x) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) < 0, \quad x \in \mathcal{Q}\backslash\mathcal{E}. \tag{8.1}$$

If every equilibrium point of (3.1) is Lyapunov stable in probability, then (3.1) is stochastically semistable. Moreover, if $\mathcal{Q} = \mathbb{R}^n$ and $V(x) \to \infty$ as $\|x\| \to \infty$, then (3.1) is globally stochastically semistable.

Proof. Since every equilibrium point of (3.1) is Lyapunov stable in probability by assumption, for every $z \in \mathcal{E}$, there exists an open neighborhood $\mathcal{V}_z$ of z such that $s([0,\infty) \times \mathcal{V}_z \cap \mathcal{B}_\varepsilon(z))$, $\varepsilon > 0$, is bounded and contained in $\mathcal{Q}$ as $\varepsilon \to 0$. The set $\mathcal{V}_\varepsilon \triangleq \bigcup_{z\in\mathcal{E}} \mathcal{V}_z \cap \mathcal{B}_\varepsilon(z)$, $\varepsilon > 0$, is an open neighborhood of $\mathcal{E}$ contained in $\mathcal{Q}$. Consider $x \in \mathcal{V}_\varepsilon$ so that there exists $z \in \mathcal{E}$ such that $x \in \mathcal{V}_z \cap \mathcal{B}_\varepsilon(z)$ and $s(t,x) \in \mathcal{H}_n^{\mathcal{V}_z\cap\mathcal{B}_\varepsilon(z)}$, $t \ge 0$, as $\varepsilon \to 0$. Since $\mathcal{V}_z \cap \mathcal{B}_\varepsilon(z)$ is bounded and invariant with respect to the solution of (3.1) as $\varepsilon \to 0$, it follows that $\mathcal{V}_\varepsilon$ is invariant with respect to the solution of (3.1) as $\varepsilon \to 0$. Furthermore, it follows from (8.1) that $\mathcal{L}V(s(t,x)) < 0$, $t \geq 0$, and hence, since $\mathcal{V}_\varepsilon$ is bounded it follows from Corollary 4.1 of [87] that $\lim_{t\to\infty} \mathcal{L}V(s(t,x)) \overset{\text{a.s.}}{=} 0$ as $\varepsilon \to 0$.

It is easy to see that $\mathcal{L}V(x) \neq 0$ by assumption and $\mathcal{L}V(x_{\text{e}}) = 0$, $x_{\text{e}} \in \mathcal{E}$. Therefore, $s(t,x) \overset{\text{a.s.}}{\to} \mathcal{E}$ as $t \to \infty$ and $\varepsilon \to 0$, which implies that $\lim_{\text{dist}(x,\mathcal{E})\to 0} \mathbb{P}^x(\lim_{t\to\infty} \text{dist}(s(t,x),\mathcal{E}) = 0) = 1$. Finally, since every point in $\mathcal{E}$ is Lyapunov stable in probability, it follows from proposition 8.1 that $\lim_{t\to\infty} s(t,x) \overset{\text{a.s.}}{=} x^*$ as $x \to x^*$, where $x^* \in \mathcal{E}$ is Lyapunov stable in probability. Hence, by Definition 8.3, (3.1) is semistable.

Finally, for $\mathcal{Q} = \mathbb{R}^n$ global stochastic semistability follows from identical arguments using the radially unbounded condition on $V(\cdot)$. $\square$

Finally, we provide a partial converse to Theorem 8.1. For this result, recall that $\mathcal{L}V(x_{\text{e}}) = 0$ for every $x_{\text{e}} \in \mathcal{E}$. Also note that it follows from (3.6) that $\mathcal{L}V(x) = \mathcal{L}V(s(0,x))$. In addition, the following definition is required.

Definition 8.4. For a given $\rho \in (0,1)$, the *ρ-domain of semistability* is the set of points $x_0 \in \mathcal{D} \subseteq \mathbb{R}^n$ such that if $x(t)$, $t \geq 0$, is a solution to (3.1) with $x(0) \overset{\text{a.s.}}{=} x_0$, then $x(t)$ converges to a Lyapunov stable in probability equilibrium point in $\mathcal{D}$ with probability greater than or equal to $1-\rho$.

Theorem 8.2. Consider the nonlinear stochastic dynamical system (3.1). Suppose (3.1) is stochastically semistable with a ρ-domain of semistability $\mathcal{D}_0$. Then there exist a continuous nonnegative function $V : \mathcal{D}_0 \to \overline{\mathbb{R}}_+$ and a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that *i)* $V(x) = 0$, $x \in \mathcal{E}$, *ii)* $V(x) \geq \alpha(\text{dist}(x,\mathcal{E}))$, $x \in \mathcal{D}_0$, and *iii)* $\mathcal{L}V(x) < 0$, $x \in \mathcal{D}_0 \backslash \mathcal{E}$.

Proof. Let $\mathfrak{B}^{x_0}$ denote the set of all sample trajectories of (3.1) for which $\lim_{t\to\infty} \text{dist}(x(t,\omega),\mathcal{E}) = 0$ and $x(\{t \geq 0\},\omega) \in \mathfrak{B}^{x_0}$, $\omega \in \Omega$, and let $\mathbb{1}_{\mathfrak{B}^{x_0}}(\omega)$, $\omega \in \Omega$, denote the indicator function defined on the set $\mathfrak{B}^{x_0}$, that is,

$$\mathbb{1}_{\mathfrak{B}^{x_0}}(\omega) \triangleq \begin{cases} 1, & \text{if } x(\{t \geq 0\}, \omega) \in \mathfrak{B}^{x_0}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that by definition $\mathbb{P}^{x_0}(\mathfrak{B}^{x_0}) \geq 1 - \rho$ for all $x_0 \in \mathcal{D}_0$. Define the function $V : \mathcal{D}_0 \to \overline{\mathbb{R}}_+$ by

$$V(x) \triangleq \sup_{t \geq 0} \left\{ \frac{1+2t}{1+t} \mathbb{E}\left[\operatorname{dist}(s(t,x),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{x}}(\omega)\right] \right\}, \quad x \in \mathcal{D}_0, \tag{8.2}$$

and note that $V(\cdot)$ is well defined since (3.1) is stochastically semistable. Clearly, (i) holds. Furthermore, since $V(x) \geq \operatorname{dist}(x, \mathcal{E})$, $x \in \mathcal{D}_0$, it follows that (ii) holds with $\alpha(r) = r$.

To show that $V(\cdot)$ is continuous on $\mathcal{D}_0 \backslash \mathcal{E}$, define $T : \mathcal{D}_0 \backslash \mathcal{E} \to [0, \infty)$ by $T(z) \triangleq \inf\{h : \mathbb{E}\left[\operatorname{dist}(s(h,z),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{z}}(\omega)\right] < \operatorname{dist}(z,\mathcal{E})/2$ for all $t \geq h > 0\}$, and denote

$$\mathcal{W}_\varepsilon \triangleq \left\{ x \in \mathcal{D}_0 : \mathbb{P}^{x}\left(\sup_{t \geq 0} \operatorname{dist}(s(t,x),\mathcal{E}) \leq \varepsilon \right) \geq 1 - \rho \right\}. \tag{8.3}$$

Note that $\mathcal{W}_\varepsilon \supset \mathcal{E}$ is open and contains an open neighborhood of $\mathcal{E}$. Consider $z \in \mathcal{D}_0 \backslash \mathcal{E}$ and define $\lambda \triangleq \operatorname{dist}(z, \mathcal{E}) > 0$. Then it follows from stochastic semistability of (3.1) that there exists $h > 0$ such that $\mathbb{P}^{z}\left(s(h,z) \in \mathcal{W}_{\lambda/2}\right) \geq 1 - \rho$. Consequently, $\mathbb{P}^{z}\left(s(h+t,z) \in \mathcal{W}_{\lambda/2}\right) \geq 1 - \rho$ for all $t \geq 0$, and hence, it follows that $T(z)$ is well defined. Since $\mathcal{W}_{\lambda/2}$ is open, there exists a neighborhood $\mathcal{B}_\sigma(s(T(z),z)$ such that $\mathbb{P}^{z}\left(\mathcal{B}_\sigma(s(T(z),z)) \subset \mathcal{W}_{\lambda/2}\right) \geq 1 - \rho$. Hence, $\mathcal{N} \subset \mathcal{D}_0$ is a neighborhood of z such that $s_{T(z)}(\mathcal{H}_n^{\mathcal{N}}) \triangleq \mathcal{B}_\sigma(s(T(z),z))$.

Next, choose $\eta > 0$ such that $\eta < \lambda/2$ and $\mathcal{B}_\eta(z) \subset \mathcal{N}$. Then, for every $t > T(z)$ and $y \in \mathcal{B}_\eta(z)$,

$$[(1+2t)/(1+t)]\mathbb{E}\left[\operatorname{dist}(s(t,y),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{y}}(\omega)\right] \leq 2\mathbb{E}\left[\operatorname{dist}(s(t,y),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{y}}(\omega)\right] \leq \lambda.$$

Therefore, for every $y \in \mathcal{B}_\eta(z)$,

$$\begin{aligned} V(z) - V(y) = & \sup_{t \geq 0} \left\{ \frac{1+2t}{1+t} \mathbb{E}\left[\operatorname{dist}(s(t,z),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{z}}(\omega)\right] \right\} \\ & - \sup_{t \geq 0} \left\{ \frac{1+2t}{1+t} \mathbb{E}\left[\operatorname{dist}(s(t,y),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{y}}(\omega)\right] \right\} \end{aligned}$$

$$
\begin{aligned}
&= \sup_{0\le t\le T(z)} \left\{ \frac{1+2t}{1+t} \mathbb{E}\left[\operatorname{dist}(s(t,z),\mathcal{E})\mathbb{1}_{\mathfrak{B}^z}(\omega)\right] \right\} \\
&\quad - \sup_{0\le t\le T(z)} \left\{ \frac{1+2t}{1+t} \mathbb{E}\left[\operatorname{dist}(s(t,y),\mathcal{E})\mathbb{1}_{\mathfrak{B}^y}(\omega)\right] \right\}. \qquad (8.4)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|V(z)-V(y)| &\le \sup_{0\le t\le T(z)} \left| \frac{1+2t}{1+t} \big(\mathbb{E}\left[\operatorname{dist}(s(t,z),\mathcal{E})\mathbb{1}_{\mathfrak{B}^z}(\omega)\right] \right. \\
&\qquad \left. - \mathbb{E}\left[\operatorname{dist}(s(t,y),\mathcal{E})\mathbb{1}_{\mathfrak{B}^y}(\omega)\right] \big) \right| \\
&\le 2 \sup_{0\le t\le T(z)} \left| \mathbb{E}\left[\operatorname{dist}(s(t,z),\mathcal{E})\mathbb{1}_{\mathfrak{B}^z}(\omega)\right] - \mathbb{E}\left[\operatorname{dist}(s(t,y),\mathcal{E})\mathbb{1}_{\mathfrak{B}^y}(\omega)\right] \right| \\
&\le 2 \sup_{0\le t\le T(z)} \mathbb{E}\left[\operatorname{dist}(s(t,z),s(t,y))\right], \quad z\in\mathcal{D}_0\backslash\mathcal{E}, \quad y\in\mathcal{B}_\eta(z). \qquad (8.5)
\end{aligned}
$$

Now, since $f(\cdot)$ and $D(\cdot)$ satisfy (3.3) and (3.4), it follows from continuous dependence of solutions $s(\cdot,\cdot)$ on system initial conditions ([5], Theorem 7.3.1) and (8.5) that $V(\cdot)$ is continuous on $\mathcal{D}_0\backslash\mathcal{E}$.

To show that $V(\cdot)$ is continuous on $\mathcal{E}$, consider $x_{\mathrm{e}}\in\mathcal{E}$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{D}_0\backslash\mathcal{E}$ that converges to x_{e}. Since x_{e} is Lyapunov stable in probability with, it follows that $x(t) \overset{\text{a.s.}}{\equiv} x_{\mathrm{e}}$ is the unique solution to (3.1) with $x(0) \overset{\text{a.s.}}{=} x_{\mathrm{e}}$. By continuous dependence of solutions $s(\cdot,\cdot)$ on system initial conditions ([5], Theorem 7.3.1), $s(t,x_n) \overset{\text{a.s.}}{\to} s(t,x_{\mathrm{e}}) \overset{\text{a.s.}}{=} x_{\mathrm{e}}$ as $n\to\infty$, $t\ge 0$.

Let $\varepsilon>0$ and note that it follows from *ii*) of proposition 2.2 in [112] that there exists $\delta=\delta(x_{\mathrm{e}})>0$ such that for every solution of (3.1) in $\mathcal{B}_\delta(x_{\mathrm{e}})$ there exists $\hat{T}=\hat{T}(x_{\mathrm{e}},\varepsilon)>0$ such that $\mathbb{P}\left(s_t(\mathcal{H}_n^{\mathcal{B}_\delta(x_{\mathrm{e}})})\subset\mathcal{W}_\varepsilon\right)\ge 1-\rho$ for all $t\ge\hat{T}$. Next, note that there exists a positive integer N_1 such that $x_n\in\mathcal{B}_\delta(x_{\mathrm{e}})$ for all $n\ge N_1$. Now, it follows from (8.2) that

$$
V(x_n) \le 2 \sup_{0\le t\le\hat{T}} \mathbb{E}[\operatorname{dist}(s(t,x_n),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{x_n}}(\omega)] + 2\varepsilon, \quad n\ge N_1. \qquad (8.6)
$$

Next, it follows from ([5], Theorem 7.3.1) that $\mathbb{E}[|s(\cdot,x_n)|]$ converges to $\mathbb{E}[|s(\cdot,x_{\mathrm{e}})|]$ uniformly on $[0,\hat{T}]$. Hence,

$$
\lim_{n\to\infty} \sup_{0\le t\le\hat{T}} \mathbb{E}\left[\operatorname{dist}(s(t,x_n),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{x_n}}(\omega)\right] = \sup_{0\le t\le\hat{T}} \mathbb{E}\left[\lim_{n\to\infty} \operatorname{dist}(s(t,x_n),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{x_n}}(\omega)\right]
$$

$$
\begin{aligned}
&\leq \sup_{0\leq t\leq \hat{T}} \operatorname{dist}(x_{\rm e}, \mathcal{E}) \\
&= 0,
\end{aligned} \tag{8.7}
$$

which implies that there exists a positive integer $N_2 = N_2(x_{\rm e}, \varepsilon) \geq N_1$ such that

$$
\sup_{0\leq t\leq \hat{T}} \mathbb{E}\left[\operatorname{dist}(s(t, x_n), \mathcal{E})\mathbb{1}_{\mathfrak{B}^{x_n}}(\omega)\right] < \varepsilon
$$

for all $n \geq N_2$. Combining (8.6) with the above result yields $V(x_n) < 4\varepsilon$ for all $n \geq N_2$, which implies that $\lim_{n\to\infty} V(x_n) = 0 = V(x_{\rm e})$.

Finally, we show that $\mathcal{L}V(x(t))$ is negative along the solution of (3.1) on $\mathcal{D}_0\backslash\mathcal{E}$. Note that for every $x \in \mathcal{D}_0\backslash\mathcal{E}$ and $0 < h \leq 1/2$ such that $\mathbb{P}\left(s(h, x) \in \mathcal{D}_0\backslash\mathcal{E}\right) \geq 1 - \rho$, it follows from the definition of $T(\cdot)$ that $\mathbb{E}\left[V(s(h, x))\right]$ is reached at some time $\hat{t}$ such that $0 \leq \hat{t} \leq T(x)$. Hence, it follows from the law of iterated expectation that

$$
\begin{aligned}
\mathbb{E}\left[V(s(h, x))\right] &= \mathbb{E}\left[\mathbb{E}\left[\operatorname{dist}(s(\hat{t}+h, x), \mathcal{E})\mathbb{1}_{\mathfrak{B}^{s(h,x)}}(\omega)\right]\frac{1+2\hat{t}}{1+\hat{t}}\right] \\
&= \mathbb{E}\left[\operatorname{dist}(s(\hat{t}+h, x), \mathcal{E})\mathbb{1}_{\mathfrak{B}^{x}}(\omega)\right]\frac{1+2\hat{t}+2h}{1+\hat{t}+h}\left[1 - \frac{h}{(1+2\hat{t}+2h)(1+\hat{t})}\right] \\
&\leq V(x)\left[1 - \frac{h}{2(1+T(x))^2}\right],
\end{aligned} \tag{8.8}
$$

which implies that

$$
\mathcal{L}V(x) = \lim_{h\to 0^+} \frac{\mathbb{E}\left[V(s(h, x))\right] - V(x)}{h} \leq -\frac{1}{2}V(x)(1+T(x))^{-2} < 0, \quad x \in \mathcal{D}_0\backslash\mathcal{E},
$$

and hence, (iii) holds. □

8.3. Stochastic Finite Time Semistability

In this section, we extend the results of Section 8.2 to address *stochastic finite-time semistability*. Here we assume that the uniform Lipschitz continuity condition (3.3) and the growth condition (3.4) are satisfied for all x, $y \in \mathcal{D}\backslash\mathcal{E}$. Furthermore, we assume that for every initial condition $x_0 \in \mathcal{D}\backslash\mathcal{E}$, (3.1) has a unique solution in forward time.

The notion of stochastic finite time semistability involves finite time almost sure convergence along with stochastic semistability.

Definition 8.5. An equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\mathrm{e}} \in \mathcal{E}$ of (3.1) is (*globally*) *stochastically finite-time semistable* if there exists an operator $T : \mathcal{H}_n \to \mathcal{H}_1^{[0,\infty)}$, called the *stochastic settling-time operator*, such that the following statements hold.

i) *Finite-time convergence in probability.* For every $x(0) \in \mathcal{H}_n \backslash \mathcal{E}$, $s^{x(0)}(t)$ is defined on $[0, T(x(0)))$, $s^{x(0)}(t) \in \mathcal{H}_n \backslash \mathcal{E}$ for all $t \in [0, T(x(0)))$, and

$$\mathbb{P}^{x_0}\left(\lim_{t \to T(x(0))} \operatorname{dist}(s^{x(0)}(t), \mathcal{E}) = 0\right) = 1.$$

ii) *Lyapunov stability in probability.* For every $\varepsilon > 0$,

$$\lim_{x_0 \to x_{\mathrm{e}}} \mathbb{P}^{x_0}\left(\sup_{0 \le t < \infty} \|s^{x(0)}(t) - x_{\mathrm{e}}\| > \varepsilon\right) = 0.$$

Equivalently, for every $\varepsilon > 0$ and $\rho \in (0, 1)$, there exist $\delta = \delta(\varepsilon, \rho) > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(x_{\mathrm{e}})$, $\mathbb{P}^{x_0}\left(\sup_{0 \le t < \infty} \|s^{x(0)}(t) - x_{\mathrm{e}}\| > \varepsilon\right) \le \rho$.

iii) *Finiteness of the stochastic settling-time operator.* For every $x \in \mathcal{H}_n \backslash \mathcal{E}$ the stochastic settling-time operator $T(x)$ exists and is finite with probability one, that is, $\mathbb{E}^x\left[T(x)\right] < \infty$. The dynamical system (3.1) is (*globally*) *stochastically finite time semistable* if every equilibrium solution of (3.1) is globally stochastically finite time semistable.

It is easy to see from Definition 8.5 that

$$T(x(0)) = \inf\{t \in \overline{\mathbb{R}}_+ : s(t, x(0)) = 0\}, \quad x(0) \in \mathcal{H}_n^{\mathbb{R}^n \backslash \mathcal{E}}.$$

Proposition 8.2. Suppose (3.1) is stochastically finite time semistable. Let $x_{\mathrm{e}} \in \mathcal{E}$ be an equilibrium point of (3.1) and let $T : \mathcal{H}_n \to \mathcal{H}_1^{[0,\infty]}$ be the stochastic finite time operator. Then the following statements hold.

i) If $\tau \ge 0$ and $x(0) \in \mathcal{H}_n$, then $T(s(\tau, x(0))) \overset{\text{a.s.}}{=} \max\{T(x(0)) - \tau, 0\}$.

ii) $T(\cdot)$ is sample continuous on $\mathcal{H}_n$ if and only if $T(\cdot)$ is sample continuous at every $z_{\rm e} \in \mathcal{H}_n \cap \mathcal{E}$.

Proof. The proof is similar to the proof of proposition 3.2 of [116] and, hence, is omitted. $\square$

Next, we present a sufficient condition for global stochastic finite time semistability.

Theorem 8.3. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (3.1) with $\mathcal{D} = \mathbb{R}^n$ and assume that there exist a radially unbounded nonnegative function $V : \mathbb{R}^n \to \overline{\mathbb{R}}_+$ and a function $\eta : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ such that $V^{-1}(0) = \mathcal{E}$, $V(x)$ is two-times continuously differentiable for all $x \in \mathbb{R}^n \backslash \mathcal{E}$, $\eta(\cdot)$ is continuously differentiable, and, for all $x \in \mathbb{R}^n \backslash \mathcal{E}$,

$$V'(x)f(x) + \frac{1}{2}{\rm tr}\ D^{\rm T}(x)V''(x)D(x) \le -\eta(V(x)), \tag{8.9}$$

$$\int_0^{\varepsilon} \frac{{\rm d}v}{\eta(v)} < \infty, \quad \varepsilon \in [0, \infty), \tag{8.10}$$

$$\eta'(v) > 0, \qquad v \ge 0. \tag{8.11}$$

If every point in the set $\mathcal{M} \triangleq \{x \in \mathcal{Q} : \eta(V(x)) = 0\}$ is Lyapunov stable in probability, then $\mathcal{G}$ is globally stochastically finite time semistable. Moreover, there exists a settling-time operator $T : \mathcal{H}_n \to \mathcal{H}_1^{[0,\infty)}$ such that

$$\mathbb{E}^{x_0}[T(x_0)] \le \int_0^{V(x_0)} \frac{{\rm d}v}{\eta(v)}, \quad x_0 \in \mathbb{R}^n. \tag{8.12}$$

Proof. It follows from (8.9) and Corollary 4.2 of [87] that $\lim_{t\to\infty} V(x(t))$ exists and is finite almost surely, and $\lim_{t\to\infty} \eta(V(s(t,x))) \overset{\rm a.s.}{=} 0$. Therefore, $s(t,x) \overset{\rm a.s.}{\to} \mathcal{M}$ as $t \to \infty$, which implies that $\lim_{{\rm dist}(x,\mathcal{M})\to 0} \mathbb{P}^x\left(\lim_{t\to\infty} {\rm dist}(s(t,x),\mathcal{M}) = 0\right) = 1$. Now, since every point in $\mathcal{M}$ is Lyapunov stable in probability, it follows from proposition 8.1 that $\lim_{t\to\infty} s(t,x) \overset{\rm a.s.}{=} x^*$ as $x \to x^*$, where $x^* \in \mathcal{M}$ is Lyapunov stable in probability. Hence, by definition, (3.1) is globally stochastically semistable. This further implies that the stochastic settling time operator $T(x)$ exists with probability one for all $x \in \mathcal{H}_n \backslash \mathcal{E}$.

Next, we show that $T(x(0))$ is finite with probability one and satisfies (8.12), and hence, $\mathbb{E}^{x_0}[T(x(0))] < \infty$. Define $T_0 \triangleq T(x(0))$ and $\alpha(V) \triangleq \int_0^V \frac{\mathrm{d}v}{\eta(v)}$, $V \in \overline{\mathbb{R}}_+$. Now, using Itô's (chain rule) formula the stochastic differential of $V(x(t))$ along the system sample trajectories $x(t)$, $t \geq 0$, is given by

$$\mathrm{d}V(x(t)) = \mathcal{L}V(x(t))\mathrm{d}t + \frac{\partial V}{\partial x}D(x(t))\mathrm{d}w(t).$$

Next, using (8.9) it follows that

$$\begin{aligned} T_0 &= \int_0^{T_0} \frac{\eta(V(x(\tau)))}{\eta(V(x(\tau)))}\mathrm{d}\tau \\ &\leq \int_0^{T_0} -\frac{\mathcal{L}V(x(\tau))}{\eta(V(x(\tau)))}\mathrm{d}\tau \\ &\leq \int_0^{T_0} -\frac{\mathrm{d}V(x(t))}{\eta(V(x(\tau)))} + \int_0^{T_0} \frac{1}{\eta(V(x(\tau)))}\frac{\partial V}{\partial x}D(x(\tau))\mathrm{d}w(\tau) \\ &= \int_0^{T_0} -\frac{\mathrm{d}\alpha(V))}{\mathrm{d}V}\mathrm{d}V(x(t)) + \int_0^{T_0} \frac{1}{\eta(V(x(\tau)))}\frac{\partial V}{\partial x}D(x(\tau))\mathrm{d}w(\tau). \end{aligned} \tag{8.13}$$

Once again, using Ito's (chain rule) formula it follows that

$$\begin{aligned} \mathrm{d}\alpha(V(x(t))) &= \left[\frac{\partial\alpha(V(x))}{\partial x}f(x(t)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x(t))\frac{\partial^2\alpha(V(x))}{\partial x^2}D(x(t))\right]\mathrm{d}t + \frac{\partial\alpha(V(x))}{\partial x}\mathrm{d}w(t) \\ &= \left[\frac{\mathrm{d}\alpha(V)}{\mathrm{d}V}\frac{\partial V(x)}{\partial x}f(x(t)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x(t))\frac{\partial}{\partial x}\left(\frac{\mathrm{d}\alpha(V)}{\mathrm{d}V}\frac{\partial V(x)}{\partial x}\right)D(x(t))\right]\mathrm{d}t \\ &\quad + \frac{\mathrm{d}\alpha(V)}{\mathrm{d}V}\frac{\partial V(x)}{\partial x}\mathrm{d}w(t) \\ &= \frac{\mathrm{d}\alpha(V)}{\mathrm{d}V}\left[\left(\frac{\partial V(x)}{\partial x}f(x(t)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x(t))\frac{\partial^2(V(x))}{\partial x^2}D(x(t))\right)\mathrm{d}t\right. \\ &\quad \left.+\frac{\partial V(x)}{\partial x}\mathrm{d}w(t)\right] + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x(t))\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm{T}}\frac{\mathrm{d}^2\alpha(V)}{\mathrm{d}V^2}\left(\frac{\partial V(x)}{\partial x}\right)D(x(t))\mathrm{d}t \\ &= \frac{\mathrm{d}\alpha(V)}{\mathrm{d}V}\mathrm{d}V(x(t)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x(t))\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm{T}}\frac{\mathrm{d}^2\alpha(V)}{\mathrm{d}V^2}\left(\frac{\partial V(x)}{\partial x}\right)D(x(t))\mathrm{d}t. \end{aligned} \tag{8.14}$$

Hence, it follows from (8.13) and (8.11) that

$$\begin{aligned} T_0 &\leq \int_0^{T_0} -\mathrm{d}\alpha(V(x(\tau))) + \int_0^{T_0} \frac{1}{\eta(V(x(\tau)))}\frac{\partial V}{\partial x}D(x(\tau))\mathrm{d}w(\tau) \\ &\quad + \int_0^{T_0} \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x(\tau))\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm{T}}\frac{\mathrm{d}^2\alpha(V)}{\mathrm{d}V^2}\left(\frac{\partial V(x)}{\partial x}\right)D(x(\tau))\mathrm{d}\tau \end{aligned}$$

$$
\begin{aligned}
&= \alpha(V(x(0))) - \alpha(V(x(T_0))) + \int_0^{T_0} \frac{1}{\eta(V(x(\tau)))} \frac{\partial V}{\partial x} D(x(\tau))\mathrm{d}w(\tau) \\
&\quad - \int_0^{T_0} \frac{\eta'(V)}{\eta^2(V)} \frac{1}{2}\mathrm{tr}\left(\frac{\partial V(x)}{\partial x} D^{\mathrm{T}}(x(\tau))\right)^{\mathrm{T}} \left(\frac{\partial V(x)}{\partial x} D(x(\tau))\right)\mathrm{d}\tau \\
&\le \int_0^{V(x(0))} \frac{\mathrm{d}v}{\eta(v)} - \int_0^{V(x(T_0))} \frac{\mathrm{d}v}{\eta(v)} + \int_0^{T_0} \frac{1}{\eta(V(x(\tau)))} \frac{\partial V}{\partial x} D(x(\tau))\mathrm{d}w(\tau). \qquad (8.15)
\end{aligned}
$$

Taking the expectation on both sides of (8.15) and using the fact that $x(0) \overset{\text{a.s.}}{=} x_0$ and $\mathbb{P}^{x_0}(x(T_0) \in \mathcal{E}) = 1$ implies $V(x(T_0)) \overset{\text{a.s.}}{=} 0$, (8.12) follows. □

If $\eta(V) = cV^{\theta}$, where $c > 0$ and $\theta \in (0, 1)$, then $\eta(\cdot)$ satisfies (8.10) and (8.11). In this case, (8.12) becomes

$$
\mathbb{E}^{x_0}[T(x(0))] \le \frac{V(x_0)^{1-\theta}}{c(1-\theta)}.
$$

8.4. Almost Sure Asymptotic Consensus for Stochastic Dynamical Networks

In this section, we use the results of Section 8.2 to develop a thermodynamically motivated consensus framework for multiagent nonlinear stochastic systems that achieve stochastic semistability and almost sure state equipartition. Here we use graph-theoretic notions to represent a dynamical network and present solutions to the consensus problem for networks with undirected graph topologies (or information flows).

We begin by establishing some notion and definitions. Specifically, let $\mathfrak{G}(\mathcal{C}) = (\mathcal{V}, \mathcal{E})$ be a *directed graph* (or digraph) denoting the dynamical network (or dynamic graph) with the set of *nodes* (or vertices) $\mathcal{V} = \{1, \ldots, q\}$ involving a finite nonempty set denoting the agents, the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ involving a set of ordered pairs denoting the direction of information flow, and a *connectivity matrix* $\mathcal{C} \in \mathbb{R}^{q \times q}$ such that $\mathcal{C}_{(i,j)} = 1$, $i, j = 1, \ldots, q$, if $(j, i) \in \mathcal{E}$, while $\mathcal{C}_{(i,j)} = 0$ if $(j, i) \notin \mathcal{E}$. The edge $(j, i) \in \mathcal{E}$ denotes that agent j can obtain information from agent i, but not necessarily vice versa. Moreover, we assume $\mathcal{C}_{(i,i)} = 0$ for all $i \in \mathcal{V}$. A *graph* or *undirected graph* $\mathfrak{G}$ associated with the connectivity matrix $\mathcal{C} \in \mathbb{R}^{q \times q}$ is a directed graph for which the *arc set* is symmetric, that is, $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$. Weighted graphs can also be considered here; however, since this extension does not alter any of the conceptual results in the paper we do not consider this extension for simplicity of exposition.

To address the consensus problem, consider q continuous-time agents with dynamics

$$\mathrm{d}x_i(t) = u_i(t)\mathrm{d}t + \mathrm{row}_i(D(x(t)))\mathrm{d}w(t), \quad i = 1, \ldots, q, \quad x_i(0) \stackrel{\text{a.s.}}{=} x_{i0}, \quad t \geq 0, \tag{8.16}$$

where $q \geq 2$ is the number of agents in the network with a communication graph topology $\mathfrak{G}(\mathcal{C})$, $D(x)\mathrm{d}w$, where $D(x) = [\mathrm{row}_1(D(x)), \ldots, \mathrm{row}_q(D(x))]^{\mathrm{T}} : \mathbb{R}^q \to \mathbb{R}^q \times \mathbb{R}^d$, captures probabilistic variations in the information transfer rates between agents, and, for every $i \in \{1, \ldots, q\}$, $x_i(t) \in \mathcal{H}_1$ denotes the information state of the ith agent and $u_i(t) \in \mathcal{H}_1$ denotes the control input of the ith agent. For a general distributed control architecture resulting in a network consensus action corresponding to an underlying conservation law, we assume $\mathbf{e}_q^{\mathrm{T}} D(x) = 0$, $x \in \mathbb{R}^q$, where $\mathbf{e}_q \triangleq [1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^q$, and where the agent state $x_i(t) \in \mathcal{H}_1$ denotes the generalized charge (i.e., Nöether charge or simply charge) state and the control input $u_i(t) \in \mathcal{H}_1$ denotes the conserved current input for all $t \geq 0$.

The nonlinear consensus protocol is given by

$$u_i(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j(t)) - \sigma_{ji}(x_i(t))], \tag{8.17}$$

where $\sigma_{ij}(\cdot)$, $i, j \in \{1, \ldots, q\}$, $i \neq j$, are Lipschitz continuous. Here we assume that the control process $u_i(\cdot)$ in (8.17) is restricted to a class of admissible control protocols consisting of measurable functions adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that, for every $i \in \{1, \ldots, q\}$, $u_i(\cdot) \in \mathcal{H}_1$, $t \geq 0$, and, for all $t \geq s$, $w_i(t) - w_i(s)$ is independent of $u_i(\tau)$, $w_i(\tau)$, $\tau \leq s$, and $x_i(0)$, and hence, $u_i(\cdot)$ is nonanticipative. Furthermore, we assume $u_i(\cdot)$ takes values in a compact metrizable set, and hence, it follows from Theorem 2.2.4 of [4] that there exists a unique pathwise solution to (8.16) and (8.17) in $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}^{x_{i0}})$ for every $i \in \{1, \ldots, q\}$. Finally, note that the closed-loop system (8.16) and (8.17) is given by

$$\mathrm{d}x_i(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j(t)) - \sigma_{ji}(x_i(t))]\mathrm{d}t + \mathrm{row}_i(D(x(t)))\mathrm{d}w(t),$$

$$i = 1, \ldots, q, \quad x_i(0) \stackrel{\text{a.s.}}{=} x_{i0}, \quad t \geq 0. \quad (8.18)$$

Equation (8.18) represents the collective dynamics of q agents which interact by exchanging charge. The coefficients scaling the functions $\sigma_{ij}(\cdot)$, $i, j \in \{1, \ldots, q\}$, $i \neq j$, appearing in (8.18) represent the topology of the charge exchange between the agents. More specifically, given $i, j \in \{1, \ldots, q\}$, $i \neq j$, a coefficient of $\mathcal{C}_{(i,j)} = 1$ denotes that subsystem j receives charge from subsystem i, and a coefficient of zero denotes that subsystem i and j are disconnected, and hence, cannot share any charge.

Remark 8.1. Although our results can be directly extended to the case where (8.16) and (8.17) describe the dynamics of an aggregate multiagent system with an aggregate state vector $x(t) = [x_1^{\mathrm{T}}(t), \ldots, x_q^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathcal{H}_{Nq}$, where $x_i(t) \in \mathcal{H}_N$ and $u_i(t) \in \mathcal{H}_N$, $i = 1, \ldots, q$, by using Kronecker algebra, for simplicity of exposition we focus on individual agent states evolving in $\mathcal{H}_1$ (i.e., $N = 1$).

Next, note that since

$$\mathbf{e}_q^{\mathrm{T}} \mathrm{d}x(t) = \mathbf{e}_q^{\mathrm{T}} f(x(t)) \mathrm{d}t + \mathbf{e}_q^{\mathrm{T}} D(x(t)) \mathrm{d}w(t) = 0, \quad x(0) \stackrel{\text{a.s.}}{=} x_0, \quad t \geq 0, \qquad (8.19)$$

it follows that $\sum_{i=1}^{q} \mathrm{d}x_i(t) \stackrel{\text{a.s.}}{=} 0$, $t \geq 0$, which implies that the total system charge is conserved, and hence, the controlled network satisfies an underlying conservation law. Now, it follows from Nöether's theorem [47] that to every conservation law there corresponds a symmetry. To show this for our multiagent network, the following definition and assumptions are needed.

Definition 8.6 [9]. A directed graph $\mathfrak{G}(\mathcal{C})$ is *strongly connected* if for every ordered pair of vertices (i, j), $i \neq j$, there exists a *path* (i.e., a sequence of arcs) leading from i to j.

Recall that the connectivity matrix $\mathcal{C} \in \mathbb{R}^{q \times q}$ is *irreducible*, that is, there does not exist a permutation matrix such that $\mathcal{C}$ is cogredient to a lower-block triangular matrix, if and only if $\mathfrak{G}(\mathcal{C})$ is strongly connected (see Theorem 2.7 of [9]).

Assumption 8.1. For the connectivity matrix $\mathcal{C} \in \mathbb{R}^{q\times q}$ associated with the multiagent stochastic dynamical system $\mathcal{G}$ defined by

$$\mathcal{C}_{(i,j)} \triangleq \begin{cases} 0, & \text{if } \sigma_{ij}(x_j) - \sigma_{ji}(x_i) \equiv 0, \\ 1, & \text{otherwise}, \end{cases} \qquad i \neq j, \quad i,j = 1,\ldots,q, \tag{8.20}$$

and

$$\mathcal{C}_{(i,i)} \triangleq -\sum_{k=1,\,k\neq i}^{q} \mathcal{C}_{(k,i)}, \quad i = j, \quad i = 1,\ldots,q, \tag{8.21}$$

rank $\mathcal{C} = q-1$, and for $\mathcal{C}_{(i,j)} = 1$, $i \neq j$, $\sigma_{ij}(x_j) - \sigma_{ji}(x_i) = 0$ if and only if $x_i = x_j$.

Assumption 8.2. For $i,j = 1,\ldots,q$,

$$\sum_{j=1,\,j\neq i}^{q} \mathcal{C}_{(i,j)}(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \leq -\mathrm{row}_i(D(x))\mathrm{row}_i^{\mathrm{T}}(D(x)).$$

The information connectivity between the agents can be represented by the network communication graph topology $\mathfrak{G}(\mathcal{C})$ having q nodes such that $\mathfrak{G}(\mathcal{C})$ has an undirected edge from node i to node j if and only if agent j can receive charge from agent i. Since the coefficients scaling $\sigma_{ij}(\cdot)$, $i,j \in \{1,\ldots,q\}$, $i \neq j$, are constants, the communication graph topology of the network $\mathfrak{G}(\mathcal{C})$ is fixed. Furthermore, note that the graph $\mathfrak{G}$ is *weakly connected* since the underlying undirected graph is connected; that is, every agent receives charge from, or delivers charge to, at least one other agent.

The fact that $\sigma_{ij}(x_j) - \sigma_{ji}(x_i) = 0$ if and only if $x_i = x_j$, $i \neq j$, implies that agent i and j are *connected*, and hence, can share information; alternatively, $\sigma_{ij}(x_j) - \sigma_{ji}(x_i) \equiv 0$ implies that agent i and j are *disconnected*, and hence, cannot share information. Assumption 8.1 thus implies that if the charge (or generalized energies) in the connected agents i and j are equal, then charge exchange between the agents is not possible. This statement is reminiscent of the *zeroth law of thermodynamics*, which postulates that temperature equality is a necessary and sufficient condition for thermal equilibrium. Furthermore, if $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$ and rank $\mathcal{C} = q-1$, then it follows that the connectivity matrix $\mathcal{C}$ is irreducible, which implies that for any pair of i and j, $i \neq j$, of $\mathcal{G}$ there exists a sequence information connectors (information arcs) of $\mathcal{G}$ that connect agents i and j.

Assumption 8.2 implies that charge flows from charge rich agents to charge poor agents and is reminiscent of the *second law of thermodynamics*, which states that heat (i.e., energy in transition) must flow in the direction of lower temperatures. It is important to note here that due to the stochastic term $D(x)\mathrm{d}w$ capturing probabilistic variations in the charge transfer (i.e., generalized current) between the agents, the second assumption requires that the scaled net charge flow $\mathcal{C}_{(i,j)}(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)]$ is bounded by the negative intensity of the diffusion coefficient given by $\frac{1}{2}\mathrm{tr}\ D(x)D^{\mathrm{T}}(x)$. For further details on Assumptions 8.1 and 8.2, see [50, 113].

The intensity $D(x)$ of the general probabilistic variations $D(x)\mathrm{d}w$ in the agent communication can take different forms to capture communication measurement noise or errors in the information transfer rates between agents. For example, we can consider $D(x) = M\hat{D}(x)$, where

$$M \triangleq [m_{(1,2)}, \ldots, m_{(1,q)}, m_{(2,3)}, \ldots, m_{(2,q)}, \ldots, m_{(q-1,q)}] \in \mathbb{R}^{q \times \frac{1}{2}q(q-1)}$$

$$\hat{D}(x) \triangleq \mathrm{diag}[d_{(1,2)}(x), \ldots, d_{(1,q)}(x), d_{(2,3)}(x), \ldots, d_{(2,q)}(x), \ldots, d_{(q-1,q)}(x)] \in \mathbb{R}^{\frac{1}{2}q(q-1) \times \frac{1}{2}q(q-1)}$$

and $m_{(i,j)}d_{(i,j)}(x_i, x_j)\mathrm{d}w_i$ represents stochastic variations in the information flow between the ith and jth agent. Furthermore, considering

$$d_{(i,j)}(x_i, x_j) = \mathcal{C}_{(i,j)}(x_j - x_i)^p, \tag{8.22}$$

where $p > 0$ and $m_{(i,j)} \in \mathbb{R}^q$ satisfies $m_{(i,j)_i} \geq 0$, $m_{(i,j)_j} \leq 0$, $m_{(i,j)_j} = -m_{(i,j)_i}$, $m_{(i,j)_k} = 0$, $k \neq i$, $k \neq j$, where $m_{(i,j)_i}$ denotes the ith component of $m_{(i,j)}$, it follows that $\mathbf{e}_q^{\mathrm{T}} m_{(i,j)} = 0$, and hence, it can be shown that (8.19) holds. Note that (8.22) captures nonlinear relative uncertainty between interagent communication. Of course, more general nonlinear uncertainties can also be considered.

For simplicity of exposition, in the reminder of the paper we let $d = 1$ and $p = 1$, and consider q continuous-time agents with dynamics

$$\mathrm{d}x_i(t) = u_i(t)\mathrm{d}t + \sum_{j=1, j\neq i}^{q} \gamma \mathcal{C}_{(i,j)}[x_j(t) - x_i(t)]\mathrm{d}w(t), \quad i = 1, \ldots, q, \quad x_i(0) \overset{\text{a.s.}}{=} x_{i0}, \quad t \geq 0, \tag{8.23}$$

where $\gamma \in \mathbb{R}$, so that the closed-loop system (8.23) and (8.17) is given by

$$\mathrm{d}x_i(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j(t)) - \sigma_{ji}(x_i(t))]\mathrm{d}t + \sum_{j=1, j\neq i}^{q} \gamma \mathcal{C}_{(i,j)}[x_j(t) - x_i(t)]\mathrm{d}w(t),$$
$$i = 1, \ldots, q, \quad x_i(0) \overset{\text{a.s.}}{=} x_{i0}, \quad t \geq 0. \tag{8.24}$$

In this case, (8.18) can be cast in the form of (3.1) with

$$f(x) = \begin{bmatrix} \sum_{j=1, j\neq 1}^{q} \mathcal{C}_{(1,j)}[\sigma_{1j}(x_j) - \sigma_{j1}(x_1)] \\ \vdots \\ \sum_{j=1, j\neq q}^{q} \mathcal{C}_{(q,j)}[\sigma_{qj}(x_j) - \sigma_{jq}(x_q)] \end{bmatrix}, \tag{8.25}$$

$$D(x) = \begin{bmatrix} \sum_{j=1, j\neq 1}^{q} \gamma \mathcal{C}_{(1,j)}[x_j(t) - x_1(t)] \\ \vdots \\ \sum_{j=1, j\neq q}^{q} \gamma \mathcal{C}_{(q,j)}[x_j(t) - x_q(t)] \end{bmatrix}, \tag{8.26}$$

where the stochastic term $D(x)\mathrm{d}w$ represents probabilistic variations in the charge transfer rate (i.e., generalized currents) between the agents. Furthermore, Assumption 8.2 now takes the following form.

Assumption 8.2′. For $i, j = 1, \ldots, q$, $\mathcal{C}_{(i,j)}(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \leq -(q-1)\gamma^2 \mathcal{C}_{(i,j)}^2 (x_i - x_j)^2$.

Theorem 8.4. Consider the nonlinear stochastic multiagent system given by (8.18) and assume that Assumptions 8.1 and 8.2′ hold. Then, for every $\alpha \in \mathbb{R}$, $\alpha \mathbf{e}_q$ is a stochastically semistable equilibrium state of (8.18). Furthermore, $x(t) \overset{\text{a.s.}}{\to} \frac{1}{q}\mathbf{e}_q\mathbf{e}_q^{\mathrm{T}} x(0)$ as $t \to \infty$ and $\frac{1}{q}\mathbf{e}_q\mathbf{e}_q^{\mathrm{T}} x(0)$ is a stochastically semistable equilibrium state.

Proof. To show that (8.18) is stochastically semistable, first note that if $x_i = x_j$, $i, j \in \{1, \ldots, q\}$, then $f_i(x) = 0$ and $D_i(x) = 0$ for all $i = 1, \ldots, q$ is immediate from

Assumption 8.1. Next, we show that $f_i(x) = 0$ and $D_i(x) = 0$ for all $i = 1, \ldots, q$ implies $x_1 = \cdots = x_q$. If $f_i(x) = 0$ for all $i = 1, \ldots, q$, then it follows from Assumption 8.2′ that

$$\begin{aligned}
0 &= \sum_{i=1}^{q} x_i f_i(x) \\
&= \sum_{i=1}^{q} x_i \left(\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \right) \\
&= \sum_{i=1}^{q} \sum_{j=1, j\neq i}^{q} \frac{1}{2} \mathcal{C}_{(i,j)}(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \\
&\leq \sum_{i=1}^{q} \sum_{j=1, j\neq i}^{q} -\frac{q-1}{2} \gamma^2 \mathcal{C}_{(i,j)}^2 (x_i - x_j)^2 \\
&\leq 0,
\end{aligned} \tag{8.27}$$

and, by Assumption 8.2′, $\mathcal{C}_{(i,j)}(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \leq -(q-1)\gamma^2 \mathcal{C}_{(i,j)}^2 (x_i - x_j)^2 \leq 0$ for $i, j = 1, \ldots, q$. Hence, $\mathcal{C}_{(i,j)}(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] = 0$ for $i, j = 1, \ldots, q$, which implies $x_i = x_j$, $i, j = 1, \ldots, q$. Therefore, $\mathcal{E} \triangleq f^{-1}(0) \cap D^{-1}(0) = \{(x_1, \ldots, x_q) \in \mathbb{R}^q : x_1 = \cdots = x_q = \alpha, \alpha \in \mathbb{R}\}$.

Next, consider the Lyapunov function candidate

$$V(x_1, \ldots, x_q) = \sum_{i=1}^{q} \frac{1}{2}(x_i - \alpha)^2, \quad (x_1, \ldots, x_q) \in \mathbb{R}^q, \tag{8.28}$$

where $\alpha \in \mathbb{R}$. Now, the infinitesimal generator of the closed-loop system (8.18) is given by

$$\begin{aligned}
\mathcal{L}V(x_1, \ldots, x_q) = {} & \sum_{i=1}^{q} (x_i - \alpha) \left(\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \right) \\
& + \frac{1}{2} \sum_{i=1}^{q} \left(\sum_{j=1, j\neq i}^{q} \gamma \mathcal{C}_{(i,j)}(x_j - x_i) \right)^2, \quad (x_1, \ldots, x_q) \in \mathbb{R}^q.
\end{aligned} \tag{8.29}$$

Note that since $\mathcal{C}_{(i,j)} = \mathcal{C}_{(j,i)}$, $i, j \in \{1, \ldots, q\}$, $i \neq j$, and $\mathcal{C}_{(i,i)} = 0$, $i \in \{1, \ldots, q\}$, it follows that

$$-\alpha \sum_{i=1}^{q} \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] = 0, \tag{8.30}$$

and hence,

$$\sum_{i=1}^{q} (x_i - \alpha) \left(\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \right) = \sum_{i=1}^{q} x_i \left(\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \right)$$

$$= \sum_{i=1}^{q} \sum_{j=1, j\neq i}^{q} \frac{1}{2}\mathcal{C}_{(i,j)}(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)]. \tag{8.31}$$

Next, note that

$$\sum_{i=1}^{q} \left(\sum_{j=1, j\neq i}^{q} \gamma \mathcal{C}_{(i,j)}(x_j - x_i) \right)^2 \leq \sum_{i=1}^{q} (q-1) \sum_{j=1, j\neq i}^{q} \gamma^2 \mathcal{C}_{(i,j)}^2 (x_j - x_i)^2, \quad (x_1, \ldots, x_q) \in \mathbb{R}^q,$$

and hence, it follows from (8.29) that

$$\begin{aligned}
\mathcal{L}V(x_1, \ldots, x_q) &\leq \sum_{i=1}^{q} \sum_{j=1, j\neq i}^{q} \frac{1}{2}\mathcal{C}_{(i,j)}(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \\
&\quad + \frac{q-1}{2} \sum_{i=1}^{q} \sum_{j=1, j\neq i}^{q} \gamma^2 \mathcal{C}_{(i,j)}^2 (x_j - x_i)^2 \\
&= \sum_{i=1}^{q} \sum_{j=1, j\neq i}^{q} \frac{1}{2}\mathcal{C}_{(i,j)}(x_i - x_j)\Big([\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] + (q-1)\gamma^2 \mathcal{C}_{(i,j)}(x_i - x_j)\Big) \\
&\leq 0, \quad (x_1, \ldots, x_q) \in \mathbb{R}^q,
\end{aligned} \tag{8.32}$$

which, by Theorem 3.1, implies that $x_1 = \cdots = x_q = \alpha$ is Lyapunov stable in probability.

Finally, note that $\mathcal{L}V(x_1, \ldots, x_q) \neq 0$ when $x_i \neq x_j$, $i, j \in \{1, \ldots, q\}$, $i \neq j$, and hence, $\mathcal{L}V(x_1, \ldots, x_q) < 0$, $(x_1, \ldots, x_q) \in \mathbb{R}^q \backslash \mathcal{E}$. Therefore, it follows from Theorem 8.1 that $x_1 = \cdots = x_q = \alpha$ is stochastically semistable for all $\alpha \in \mathbb{R}$. Furthermore, note that $\mathbf{e}_q^{\mathrm{T}} \mathrm{d}x(t) \overset{\text{a.s.}}{=} 0$, $t \geq 0$, implies that

$$x(t) \overset{\text{a.s.}}{\to} \frac{1}{q}\mathbf{e}_q \mathbf{e}_q^{\mathrm{T}} x(0) \overset{\text{a.s.}}{=} \frac{1}{q}[x_1(0) + \cdots + x_q(0)]\mathbf{e}_q \quad \text{as } t \to \infty,$$

which proves the result. □

Example 8.1. Consider the 5 mobile agents with the communication topology shown in Figure 8.4.1 and dynamics on $\mathcal{H}_5$ given by

$$\mathrm{d}x_1(t) = u_1(t)\mathrm{d}t + \gamma[x_2(t) - x_1(t)]\mathrm{d}w(t), \quad x_1(0) \overset{\text{a.s.}}{=} x_{10}, \quad t \geq 0, \tag{8.33}$$

$$\mathrm{d}x_2(t) = u_2(t)\mathrm{d}t + \gamma[x_1(t) - x_2(t) + x_3(t) - x_2(t) + x_5(t) - x_2(t)]\mathrm{d}w(t),$$

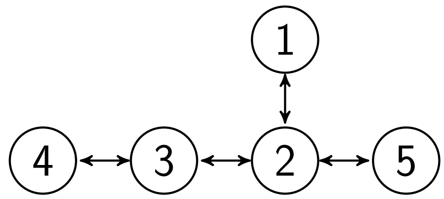


Figure 8.4.1: Communication topology for the 5 mobile agents.

$$x_2(0) \overset{\text{a.s.}}{=} x_{20}, \tag{8.34}$$

$$\mathrm{d}x_3(t) = u_3(t)\mathrm{d}t + \gamma[x_2(t) - x_3(t) + x_4(t) - x_3(t)]\mathrm{d}w(t), \quad x_3(0) \overset{\text{a.s.}}{=} x_{30}, \tag{8.35}$$

$$\mathrm{d}x_4(t) = u_4(t)\mathrm{d}t + \gamma[x_3(t) - x_4(t)]\mathrm{d}w(t), \quad x_4(0) \overset{\text{a.s.}}{=} x_{40}, \tag{8.36}$$

$$\mathrm{d}x_5(t) = u_5(t)\mathrm{d}t + \gamma[x_2(t) - x_5(t)]\mathrm{d}w(t), \quad x_5(0) \overset{\text{a.s.}}{=} x_{50}, \tag{8.37}$$

with controls

$$u_1(t) = x_2(t) - x_1(t), \tag{8.38}$$

$$u_2(t) = x_1(t) - x_2(t) + x_3(t) - x_2(t) + x_5(t) - x_2(t), \tag{8.39}$$

$$u_3(t) = x_2(t) - x_3(t) + x_4(t) - x_3(t), \tag{8.40}$$

$$u_4(t) = x_3(t) - x_4(t), \tag{8.41}$$

$$u_5(t) = x_2(t) - x_5(t). \tag{8.42}$$

Note that (8.38)–(8.42) are of the form of (8.17) with $\sigma_{ij}(x_j) = x_j$, $i, j \in \{1, 2, 3, 4, 5\}$, $i \neq j$. For our simulation we take $x_{10} = 0$, $x_{20} = 10$, $x_{30} = 20$, $x_{40} = 30$, $x_{50} = 40$, and $\gamma = 0.2$. Figure 8.4.2 shows the sample trajectories along with the standard deviation of the states of each agent versus time for 10 sample paths. The mean control profile is also plotted in Figure 8.4.2. $\triangle$

8.5. Finite Time Consensus for Stochastic Dynamical Networks

Since in many consensus control protocol applications it is desirable for the closed-loop dynamical system that exhibits semistability to also possess the property that the system

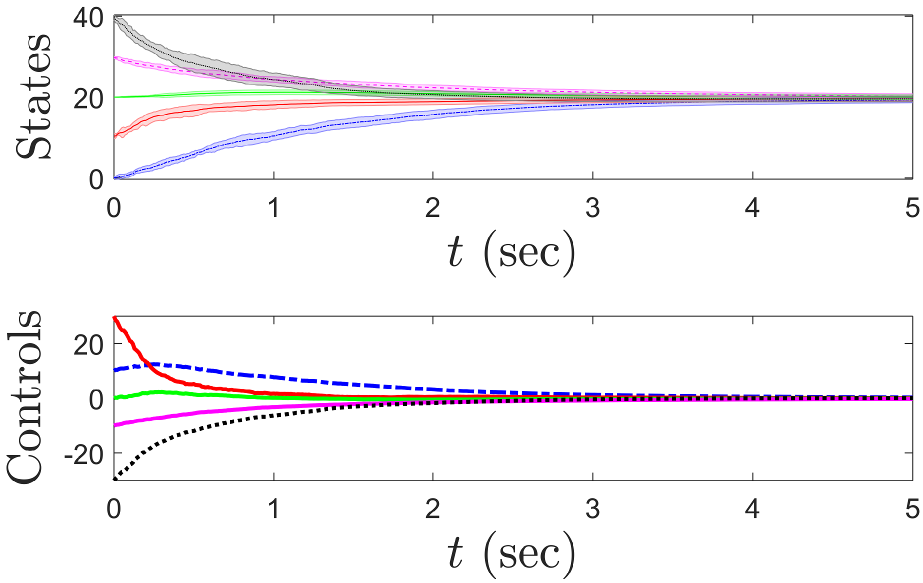


Figure 8.4.2: Sample average along with the sample standard deviation of the closed-loop system trajectories versus time; $x_1(t)$ in blue, $x_2(t)$ in red, $x_3(t)$ in green, $x_4(t)$ in magenta, and $x_5(t)$ in black. The control profile is plotted as the mean of the 10 sample runs.

trajectories that almost surely converge to a Lyapunov stable in probability system state do so in finite time rather than merely asymptotically, in this section we build on the deterministic results of [60, 144] and use Theorem 8.3 to develop a thermodynamically motivated finite time consensus framework for multiagent nonlinear stochastic systems that achieve finite time stochastic semistability and almost sure state equipartition.

Specifically, consider the q continuous-time agents with dynamics given by (8.16) with the nonlinear consensus protocol

$$u_i(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j(t)) - \sigma_{ji}(x_i(t))] + c \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} \mathrm{sign}(x_j(t) - x_i(t))|x_j(t) - x_i(t)|^{\theta}, \tag{8.43}$$

where $c > 0$ is a design constant, $0 < \theta < 1$, $\mathrm{sign}(y) \triangleq y/|y|$, $y \neq 0$, with $\mathrm{sign}(0) \triangleq 0$, and $\sigma_{ij}(\cdot)$, $i, j \in \{1, \ldots, q\}$, $i \neq j$, are as in (8.17). Note that the closed-loop system (8.16) and (8.43) is given by

$$\mathrm{d}x_i(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j(t)) - \sigma_{ji}(x_i(t))]\mathrm{d}t + c \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} \mathrm{sign}(x_j(t) - x_i(t))|x_j(t) - x_i(t)|^{\theta}$$

$$+\sum_{j=1, j\neq i}^{q} \gamma \mathcal{C}_{(i,j)}[x_j(t) - x_i(t)]\mathrm{d}w(t), \quad i = 1, \ldots, q, \quad x_i(0) \overset{\text{a.s.}}{=} x_{i0}, \quad t \geq 0. \tag{8.44}$$

Note that with $n = d = q$, (8.44) can be cast in the form of (3.1) with

$$f(x) = \begin{bmatrix} \sum_{j=1,j\neq 1}^{q} \mathcal{C}_{(1,j)}[\sigma_{1j}(x_j) - \sigma_{j1}(x_1)] + c\sum_{j=1,j\neq 1}^{q} \mathcal{C}_{(1,j)}\mathrm{sign}(x_j - x_1)|x_j - x_1|^{\theta} \\ \vdots \\ \sum_{j=1,j\neq q}^{q} \mathcal{C}_{(q,j)}[\sigma_{qj}(x_j) - \sigma_{jq}(x_q)] + c\sum_{j=1,j\neq q}^{q} \mathcal{C}_{(q,j)}\mathrm{sign}(x_j - x_q)|x_j - x_q|^{\theta} \end{bmatrix}$$

and $D(x)$ defined as in Section 8.4. Furthermore, note that since

$$\mathbf{e}_q^{\mathrm{T}}\mathrm{d}x(t) = \mathbf{e}_q^{\mathrm{T}} f(x(t))\mathrm{d}t + \mathbf{e}_q^{\mathrm{T}} D(x(t))\mathrm{d}w(t) = 0, \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0,$$

it follows that $\sum_{i=1}^{q} \mathrm{d}x_i(t) \overset{\text{a.s.}}{=} 0$, $t \geq 0$, which implies that the total system charge is conserved, and hence, the controlled network satisfies an underlying conservation law.

The following proposition is necessary for the main result in this section. For the statement of this result and the main result of this section, let $L(\mathcal{C}) = [L_{(i,j)}]$ denote the graph Laplacian of $\mathfrak{G}(\mathcal{C})$, where $\mathcal{C} = [\mathcal{C}_{(i,j)}]$ and

$$L_{(i,j)} \triangleq \begin{cases} \sum_{k=1,k\neq i}^{q} \mathcal{C}_{(i,k)}, & j = i, \\ -\mathcal{C}_{(i,j)}, & j \neq i. \end{cases} \tag{8.45}$$

Furthermore, let $\lambda_i(L(\mathcal{C}))$, $i \in \{1, \ldots, q\}$, denote the ith eigenvalue of $L(\mathcal{C})$ with $\lambda_{\min}(L(\mathcal{C})) \triangleq \lambda_1(L(\mathcal{C})) \leq \lambda_2(L(\mathcal{C})) \leq \cdots \leq \lambda_q(L(\mathcal{C})) \triangleq \lambda_{\max}(L(\mathcal{C}))$.

Proposition 8.3 [96]. Consider the nonlinear stochastic multiagent system (8.44) with communication graph topology $\mathfrak{G}(\mathcal{C})$. Then the following statements hold.

i) $\lambda_1(L(\mathcal{C})) = 0$ with associated eigenvector $\mathbf{e}_q$.

ii) $x^{\mathrm{T}}L(\mathcal{C})x = \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}(x_j - x_i)^2$ for every $x = [x_1, \ldots, x_q]^{\mathrm{T}}$, and hence, $L(\mathcal{C})$ is nonnegative definite.

iii) $\lambda_2(L(\mathcal{C})) > 0$ and

$$\lambda_2(L(\mathcal{C})) = \min_{x\neq 0,\ \mathbf{e}_q^{\mathrm{T}}x=0} \frac{x^{\mathrm{T}}L(\mathcal{C})x}{x^{\mathrm{T}}x}. \tag{8.46}$$

Hence, if $\mathbf{e}_q^{\mathrm{T}} x = 0$, then

$$x^{\mathrm{T}} L(\mathcal{C}) x \geq \lambda_2(L(\mathcal{C})) x^{\mathrm{T}} x, \quad x \in \mathbb{R}^q. \tag{8.47}$$

Theorem 8.5. Consider the nonlinear stochastic multiagent system given by (8.44) with $c > 0$ and $\theta \in (0,1)$, and assume that Assumptions 8.1 and 8.2′ hold. Then, for every $\alpha \in \mathbb{R}$, $\alpha \mathbf{e}_q$ is a stochastically finite time semistable equilibrium state of (8.44). Moreover, $x(t) = \frac{1}{q}\mathbf{e}_q \mathbf{e}_q^{\mathrm{T}} x(0)$ for all $t \geq T(x(0))$, where

$$\mathbb{E}^{x_0}[T(x(0))] \leq \frac{4V(x_0)^{\frac{1-\theta}{2}}}{c(1-\theta)(4\lambda_2(L(\mathcal{C})))^{\frac{1+\theta}{2}}}$$

and

$$V(x_0) = \frac{1}{2}(x_0 - \frac{1}{q}\mathbf{e}_q \mathbf{e}_q^{\mathrm{T}} x_0)^{\mathrm{T}} (x_0 - \frac{1}{q}\mathbf{e}_q \mathbf{e}_q^{\mathrm{T}} x_0).$$

Proof. To show that (8.44) is stochastically semistable, first note that if $x_i = x_j$, $i, j \in \{1, \ldots, q\}$, then $f_i(x) = 0$ and $D_i(x) = 0$ for all $i = 1, \ldots, q$ is immediate from Assumption 8.1. Next, we show that $f_i(x) = 0$ and $D_i(x) = 0$ for all $i = 1, \ldots, q$ implies $x_1 = \cdots = x_q$. If $f_i(x) = 0$ for all $i = 1, \ldots, q$, then it follows that from Assumption 8.2′ that

$$\begin{aligned}
0 &= \sum_{i=1}^{q} x_i f_i(x) \\
&= \sum_{i=1}^{q} x_i \left(\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \right) \\
&\quad + c \sum_{i=1}^{q} x_i \left(\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} \mathrm{sign}(x_j - x_i)|x_j - x_i|^{\theta} \right) \\
&\leq \sum_{i=1}^{q} x_i \left(\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \right) \\
&= \sum_{i=1}^{q} \sum_{j=1, j\neq i}^{q} \frac{1}{2}\mathcal{C}_{(i,j)}(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \\
&\leq \sum_{i=1}^{q} \sum_{j=1, j\neq i}^{q} -\frac{q-1}{2}\gamma^2 \mathcal{C}_{(i,j)}^2 (x_i - x_j)^2
\end{aligned}$$

$$\leq 0, \tag{8.48}$$

and, by Assumption 8.2$'$, $\mathcal{C}_{(i,j)}(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \leq -(q-1)\gamma^2\mathcal{C}_{(i,j)}^2(x_i - x_j)^2 \leq 0$ for $i, j = 1, \ldots, q$. Hence, $\mathcal{C}_{(i,j)}(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] = 0$ for $i, j = 1, \ldots, q$, which implies $x_i = x_j$, $i, j = 1, \ldots, q$. Therefore, $\mathcal{E} \triangleq f^{-1}(0) \cap D^{-1}(0) = \{(x_1, \ldots, x_q) \in \mathbb{R}^q : x_1 = \cdots = x_q = \alpha, \alpha \in \mathbb{R}\}$. Furthermore, since $\sum_{i=1}^q \mathrm{d}x_i(t) \overset{\text{a.s.}}{=} 0$, $t \geq 0$, it follows that $\sum_{i=1}^q x_i(t) \overset{\text{a.s.}}{=} \sum_{i=1}^q x_i(0)$, $t \geq 0$, and hence, $\alpha = \frac{1}{q}\mathbf{e}_q^{\mathrm{T}} x(0)$.

Next, consider the Lyapunov function candidate

$$V(x_1, \ldots, x_q) = \sum_{i=1}^q \frac{1}{2}(x_i - \alpha)^2, \quad (x_1, \ldots, x_q) \in \mathbb{R}^q, \tag{8.49}$$

where $\alpha = \frac{1}{q}\mathbf{e}_q^{\mathrm{T}} x_0$, and note that $V^{-1}(0) = \mathcal{E}$. Now, the infinitesimal generator of the closed-loop system (8.44) is given by

$$\begin{aligned}
\mathcal{L}V(x_1, \ldots, x_q) = {} & \sum_{i=1}^q (x_i - \alpha)\left(\sum_{j=1, j\neq i}^q \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)]\right) \\
& + c\sum_{i=1}^q (x_i - \alpha)\left(\sum_{j=1, j\neq i}^q \mathcal{C}_{(i,j)}\mathrm{sign}(x_j - x_i)|x_j - x_i|^\theta\right) \\
& + \frac{1}{2}\sum_{i=1}^q \left(\sum_{j=1, j\neq i}^q \gamma\mathcal{C}_{(i,j)}(x_j - x_i)\right)^2, \quad (x_1, \ldots, x_q) \in \mathbb{R}^q.
\end{aligned} \tag{8.50}$$

Using identical arguments as in the proof of Theorem 8.4, the first and last terms in (8.50) give

$$\begin{aligned}
& \sum_{i=1}^q (x_i - \alpha)\left(\sum_{j=1, j\neq i}^q \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)]\right) + \frac{1}{2}\sum_{i=1}^q \left(\sum_{j=1, j\neq i}^q \gamma\mathcal{C}_{(i,j)}(x_j - x_i)\right)^2 \\
& \quad \leq \sum_{i=1}^q \sum_{j=1, j\neq i}^q \frac{1}{2}\mathcal{C}_{(i,j)}(x_i - x_j)\Big([\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] + (q-1)\gamma^2\mathcal{C}_{(i,j)}(x_i - x_j)\Big) \\
& \quad \leq 0, \quad (x_1, \ldots, x_q) \in \mathbb{R}^q.
\end{aligned} \tag{8.51}$$

Next, the second term in (8.50) gives

$$c\sum_{i=1}^q (x_i - \alpha)\left(\sum_{j=1, j\neq i}^q \mathcal{C}_{(i,j)}\mathrm{sign}(x_j - x_i)|x_j - x_i|^\theta\right)$$

$$
\begin{aligned}
&= c\sum_{i=1}^{q} x_i \left(\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}\mathrm{sign}(x_j - x_i)|x_j - x_i|^{\theta}\right)\\
&= \frac{c}{2}\sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}(x_i - x_j)\mathrm{sign}(x_j - x_i)|x_j - x_i|^{\theta}\\
&= -\frac{c}{2}\sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} \left(\mathcal{C}_{(i,j)}^{\frac{2}{1+\theta}}(x_j - x_i)^2\right)^{\frac{1+\theta}{2}}\\
&\leq -\frac{c}{2}\left(\sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}^{\frac{2}{1+\theta}}(x_j - x_i)^2\right)^{\frac{1+\theta}{2}}, \quad (x_1, \ldots, x_q) \in \mathbb{R}^q, \qquad (8.52)
\end{aligned}
$$

where the last inequality in (8.52) follows from Fact 2.11.130 of [10]. Now, note that the last term in (8.52) satisfies

$$
-\frac{c}{2}\left(\sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}^{\frac{2}{1+\theta}}(x_j - x_i)^2\right)^{\frac{1+\theta}{2}} = -\frac{c}{2}\left(\frac{\sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}^{\frac{2}{1+\theta}}(x_j - x_i)^2}{V(x_1, \ldots, x_q)} V(x_1, \ldots, x_q)\right)^{\frac{1+\theta}{2}}. \qquad (8.53)
$$

Next, define $x_{\mathrm{s}i} \triangleq x_i - \alpha$ and note that $x_{\mathrm{s}j} - x_{\mathrm{s}i} = x_j - x_i$. Furthermore, note that $\mathbf{e}_q^{\mathrm{T}} x_{\mathrm{s}}(t) \overset{\mathrm{a.s.}}{=} 0$, $t \geq 0$, where $x_{\mathrm{s}} = [x_{\mathrm{s}1}, \ldots, x_{\mathrm{s}q}]^{\mathrm{T}}$. Now, since $\mathcal{C}_{(i,j)} = 1$ or 0, clearly $\mathcal{C}_{(i,j)}^{\frac{2}{1+\theta}} = \mathcal{C}_{(i,j)}$ for every $0 < \theta < 1$. Thus, by proposition 8.3,

$$
\begin{aligned}
\frac{\sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}^{\frac{2}{1+\theta}}(x_j - x_i)^2}{V(x_1, \ldots, x_q)} &= \frac{\sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}(x_{\mathrm{s}j} - x_{\mathrm{s}i})^2}{V(x_1, \ldots, x_q)}\\
&= \frac{2x_{\mathrm{s}}^{\mathrm{T}} L(\mathcal{C}) x_{\mathrm{s}}}{\frac{1}{2}x_{\mathrm{s}}^{\mathrm{T}} x_{\mathrm{s}}}\\
&\geq 4\lambda_2(L(\mathcal{C}))\\
&> 0, \quad (x_1, \ldots, x_q) \in \mathbb{R}^q \backslash \mathcal{E}. \qquad (8.54)
\end{aligned}
$$

Hence, using (8.51)–(8.54) it follows from (8.50) that

$$
\mathcal{L}V(x_1, \ldots, x_q) \leq -\frac{c}{2}\Big(4\lambda_2(L(\mathcal{C}))\Big)^{\frac{1+\theta}{2}} V(x_1, \ldots, x_q)^{\frac{1+\theta}{2}}, \quad (x_1, \ldots, x_q) \in \mathbb{R}^q \backslash \mathcal{E}. \qquad (8.55)
$$

Now, by Theorem 8.3, $x(t) = \frac{1}{q}\mathbf{e}_q \mathbf{e}_q^{\mathrm{T}} x(0)$, $t \geq T(x(0))$, where

$$
\mathbb{E}^{x_0}[T(x(0))] \leq \frac{4V(x_0)^{\frac{1-\theta}{2}}}{c(1-\theta)(4\lambda_2(L(\mathcal{C})))^{\frac{1+\theta}{2}}}.
$$

This completes the proof. $\square$

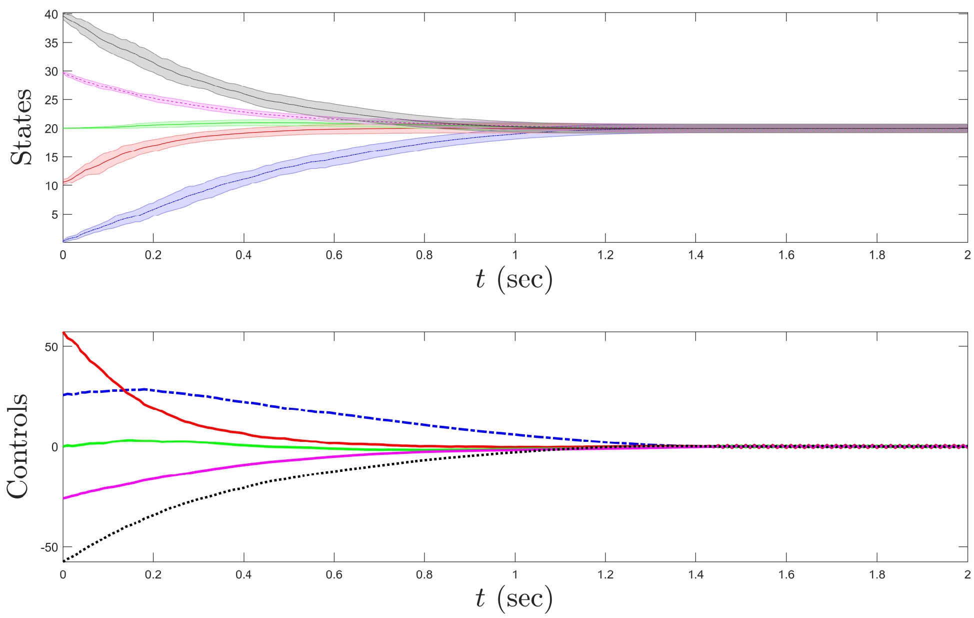


Figure 8.5.1: Sample average along with the sample standard deviation of the closed-loop system trajectories versus time; $x_1(t)$ in blue, $x_2(t)$ in red, $x_3(t)$ in green, $x_4(t)$ in magenta, and $x_5(t)$ in black. The control profile is plotted as the mean of the 10 sample runs.

Example 8.2. Consider the 5 mobile agents with the communication topology shown in Figure 8.4.1 and dynamics on $\mathcal{H}_5$ given by (8.33)–(8.37). Furthermore, let

$$u_1(t) = c\text{sign}(x_2(t) - x_1(t))|x_2(t) - x_1(t)|^{0.5} + x_2(t) - x_1(t), \tag{8.56}$$

$$u_2(t) = c \sum_{j=1,3,5} \text{sign}(x_j(t) - x_2(t))|x_j(t) - x_2(t)|^{0.5} + \sum_{j=1,3,5} (x_j(t) - x_2(t)), \tag{8.57}$$

$$u_3(t) = c \sum_{j=2,4} \text{sign}(x_j(t) - x_3(t))|x_j(t) - x_3(t)|^{0.5} + \sum_{j=2,4} (x_j(t) - x_3(t)), \tag{8.58}$$

$$u_4(t) = c\text{sign}(x_3(t) - x_4(t))|x_3(t) - x_4(t)|^{0.5} + x_3(t) - x_4(t), \tag{8.59}$$

$$u_5(t) = c\text{sign}(x_2(t) - x_5(t))|x_2(t) - x_5(t)|^{0.5} + x_2(t) - x_5(t), \tag{8.60}$$

where $c = 5$. Note that (8.56)–(8.60) are of the form of (8.43) with $\sigma_{ij}(x_j) = x_j$, $i, j \in \{1, 2, 3, 4, 5\}$, $i \neq j$. Let $x_{10} = 0$, $x_{20} = 10$, $x_{30} = 20$, $x_{40} = 30$, $x_{50} = 40$, and $\gamma = 0.2$. Figure 8.5.1 shows the sample trajectories along with the standard deviation of the states of each agent versus time for 10 sample paths. The mean control profile is also plotted in Figure 8.5.1. △

8.6. Illustrative Numerical Example

In this section, we demonstrate the proposed distributed stochastic consensus framework on a set of control commanded aircrafts achieving asymptotic pitch rate consensus. Specifically, consider the multiagent system comprised of the controlled longitudinal motion of three Boeing 747 aircrafts [17] linearized at an altitude of 40 kft and a velocity of 774 ft/sec given by

$$\dot{z}_i(t) = Az_i(t) + B\delta_i(t), \quad z_i(0) = z_{i_0}, \quad i = 1, 2, 3, \quad t \geq 0, \tag{8.61}$$

where $z_i(t) = [v_{x_i}(t), v_{z_i}(t), q_i(t), \theta_{e_i}(t)]^{\mathrm{T}} \in \mathbb{R}^4$, $t \geq 0$, is state vector of agent $i \in \{1, 2, 3\}$, with $v_{x_i}(t)$, $t \geq 0$, representing the x–body–axis component of the velocity of the aircraft center of mass with respect to the reference axes (in ft/sec), $v_{z_i}(t)$, $t \geq 0$, representing the z–body–axis component of the velocity of the aircraft center of mass with respect to the reference axes (in ft/sec), $q_i(t)$, $t \geq 0$, representing the y–body–axis component of the angular velocity of the aircraft (pitch rate) with respect to the reference axes (in crad/sec), $\theta_{e_i}(t)$, $t \geq 0$, representing the pitch Euler angle of the aircraft body axes with respect to the reference axes (in crad), $\delta(t)$, $t \geq 0$, representing the elevator control input (in crad), and

$$A = \begin{bmatrix} -0.003 & 0.039 & 0 & -0.332 \\ -0.065 & -0.319 & 7.74 & 0 \\ 0.020 & -0.101 & -0.429 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.010 \\ -0.180 \\ -1.16 \\ 0 \end{bmatrix}. \tag{8.62}$$

We propose a two-level control hierarchy composed of a lower-level controller for command following of the three aircrafts and a higher-level consensus controller for pitch rate consensus in the face of an uncertain triangular communication topology of the three aircrafts given by (8.61). To address the lower-level controller design, let $x_i(t)$, $i = 1, 2, 3$, $t \geq 0$, denote a command generated by (8.18) (i.e., the guidance command) and let $s_i(t)$, $i = 1, 2, 3$, $t \geq 0$, denote the integrator state satisfying

$$\dot{s}_i(t) = Ez_i(t) - x_i(t), \quad s_i(0) = s_{i_0}, \quad i = 1, 2, 3, \quad t \geq 0, \tag{8.63}$$

where $E = [0, 0, 1, 0]$. Now, defining the augmented state $\hat{z}(t) \triangleq [z^{\mathrm{T}}(t), s_i(t)]^{\mathrm{T}}$, (8.61) and (8.63) give

$$\dot{\hat{z}}_i(t) = \hat{A}\hat{z}_i(t) + \hat{B}_1\delta_i(t) + \hat{B}_2 x_i(t), \quad \hat{z}_i(0) = \hat{z}_{i_0}, \quad i = 1, 2, 3, \quad t \geq 0, \tag{8.64}$$

where

$$\hat{A} \triangleq \begin{bmatrix} A & 0 \\ E & 0 \end{bmatrix}, \quad \hat{B}_1 \triangleq \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \hat{B}_2 \triangleq \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \tag{8.65}$$

Furthermore, let the elevator control input be given by

$$\delta(t) = -K\hat{z}(t), \quad K = [-0.0157, 0.0831, -4.7557, -0.1400, -9.8603], \tag{8.66}$$

which is designed using an optimal linear-quadratic regulator.

For the higher-level communication consensus controller design, we use (8.17) with $\sigma_{ij}(x_j) = x_j$ and $\sigma_{ji}(x_i) = x_i$ to generate $x_i(t)$, $t \geq 0$, that has a direct effect on the lower-level controller design to achieve pitch rate consensus. Figures 8.6.1 presents the sample trajectories along with the standard deviation of the states of each agent versus time for 10 sample paths for all initial conditions set to zero and $x_1(0) \stackrel{\text{a.s.}}{=} 8$, $x_2(0) \stackrel{\text{a.s.}}{=} 4$, and $x_3(0) \stackrel{\text{a.s.}}{=} 2$. The mean control profile is also plotted in Figure 8.6.1.

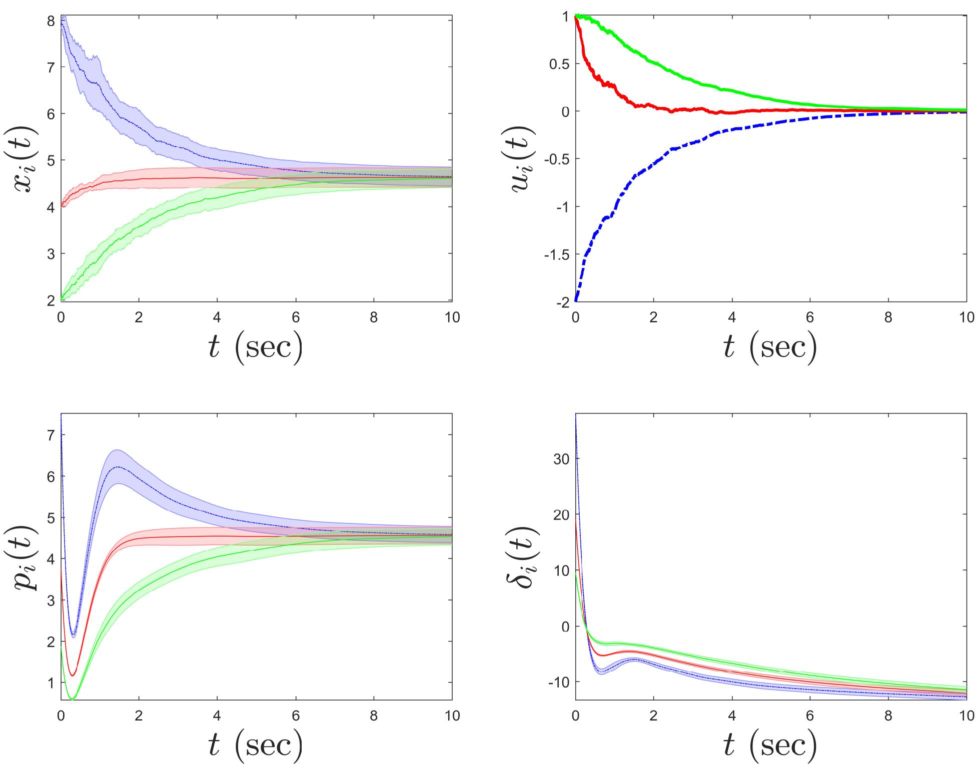


Figure 8.6.1: Sample average along with the sample standard deviation versus time for agent guidance state ($x_i(t)$, $t \geq 0$), guidance input ($u_i(t)$, $t \geq 0$), pitch rate ($q_i(t)$, $t \geq 0$), and elevator control ($\delta_i(t)$, $t \geq 0$) responses for the standard consensus protocol given by (8.17) with $k_1 = 1$. The control profile is plotted as the mean of the 10 sample runs.

Chapter 9

Conclusion and Future Research

9.1. Conclusion

Recent technological advances in communications and computation have spurred a broad interest in control law architectures involving the monitoring, coordination, integration, and operation of sensing, computing, and communication components that tightly interact with the physical processes that they control. These systems are known as cyber-physical systems and due to their use of open computation and communication platform architectures, controlled cyber-physical systems are vulnerable to adversarial attacks. In Chapter 2, we proposed a novel adaptive control architecture for addressing security and safety in cyber-physical systems. Specifically, we developed an adaptive controller that guarantees uniform ultimate boundedness of the closed-loop dynamical system in the face of adversarial sensor and actuator attacks that are time-varying and partial asymptotic stability when the sensor and actuator attacks are time-invariant.

Next, we built on this framework to develop an adaptive control algorithm for addressing security for a class of networked vehicles that comprise n human-driven vehicles sharing kinematic data and an autonomous vehicle in the aft of the vehicle formation receiving data from the preceding vehicles by wireless vehicle-to-vehicle communication devices. Specifically, we developed an adaptive controller for mitigating time-invariant, state-dependent adversarial sensor and actuator attacks while guaranteeing uniform ultimate boundedness of the closed-loop networked system.

Next, in Chapter 3, we proposed a novel adaptive control architecture for addressing security and safety in cyber-physical systems subject to exogenous disturbances. Specifically, we develop an adaptive controller for time-invariant, state-dependent adversarial sensor and actuator attacks in the face of stochastic exogenous disturbances modeled as Markov processes. We showed that the proposed controller guarantees uniform ultimate boundedness of the closed-loop dynamical system in a mean-square sense. We further discussed the practicality of the proposed approach and provided a numerical example involving the lateral directional dynamics of an aircraft to illustrate the efficacy of the proposed adaptive control architecture.

In Chapter 4, we addressed networked multiagent systems subject to stochastic exogenous disturbances with compromised sensor and actuators. First, for a class of linear leader-follower multiagent systems, we developed a new structure of the neighborhood synchronization error for the control design protocol of each follower. The proposed control algorithm addresses time-varying multiplicative sensor attacks on the leader state measurements. In addition, the framework addresses time-varying multiplicative actuator attacks on the followers that do not have a communication link with the leader and additive actuator attacks on all follower agents in the network. The proposed adaptive controller guarantees uniform ultimate boundedness of the state tracking error for each agent in a mean-square sense.

Next, we extended the approach to develop a distributed robust adaptive control architecture that can foil malicious sensor and actuator attacks in the face of exogenous stochastic disturbances and follower agent model uncertainties. Specifically, for a class of linear multiagent uncertain systems with an undirected communication graph topology we develop a neighborhood synchronization error for the distributed robust adaptive control protocol design of each follower to account for actuator and sensor attacks on the leader state as well as all of the follower agents in the network. The proposed robust adaptive controller guarantees uniform ultimate boundedness in probability of the state tracking error for each follower agent in a mean-square sense. Finally, the framework was extended to address output feedback architectures for leader-follower multiagent systems with stochastic disturbances and sensor and actuator attacks.

In Chapter 5, we developed an energy-based static and dynamic control framework for stochastic port-controlled Hamiltonian systems. In particular, we obtained constructive sufficient conditions for stochastic feedback stabilization that provide a shaped energy function for the closed-loop system while preserving a Hamiltonian structure at the closed-loop level. In the dynamic control case, energy shaping was achieved by combining the physical energy of the plant and the emulated energy of the controller.

Next, in Chapter 6, we derived stability margins for optimal and inverse optimal stochastic feedback regulators. Specifically, gain, sector, and disk margin guarantees were obtained for nonlinear stochastic dynamical systems controlled by nonlinear optimal and inverse optimal Hamilton-Jacobi-Bellman controllers that minimize a nonlinear-nonquadratic performance criterion with cross-weighting terms. Furthermore, using the newly developed notion of stochastic dissipativity we derived a return difference inequality to provide connections between stochastic dissipativity and optimality of nonlinear controllers for stochastic dynamical systems. In particular, using extended Kalman-Yakubovich-Popov conditions characterizing stochastic dissipativity we showed that our optimal feedback control law satisfies a return difference inequality predicated on the infinitesimal generator of a controlled Markov diffusion process if and only if the controller is stochastically dissipative with respect to a specific quadratic supply rate.

A constructive finite time stabilizing feedback control law was derived in Chapter 7 for stochastic dynamical systems driven by Wiener processes based on the existence of a stochastic control Lyapunov function. In addition, we presented necessary and sufficient conditions for continuity of such controllers. Moreover, using stochastic control Lyapunov functions, we constructed a universal inverse optimal feedback control law for nonlinear stochastic dynamical systems that possesses guaranteed gain and sector margins.

Finally, in Chapter 8 we focused on semistability and finite time semistability analysis and synthesis of stochastic dynamical systems having a continuum of equilibria. We extended the theories of semistability and finite-time semistability for deterministic dynamical systems to develop a rigorous framework for stochastic semistability and stochastic finite-time semistability. Specifically, Lyapunov and converse Lyapunov theorems for stochastic semistability are developed for dynamical systems driven by Markov diffusion processes. These results were then used to develop a general framework for designing semistable consensus protocols for dynamical networks in the face of stochastic communication uncertainty for achieving multiagent coordination tasks in finite time. The proposed controller architectures involved the exchange of generalized charge or energy state information between agents guaranteeing that the closed-loop dynamical network is stochastically semistable to an equipartitioned equilibrium representing a state of almost sure consensus consistent with basic thermodynamic principles.

9.2. Recommendations for Future Research

The framework in Chapters 2 and 3 can be extened to develop reliable hybrid-adaptive control architectures for cyber-physical systems involving system nonlinearities and system modeling uncertainty, with integrated verification and validation, for providing robust system performance and reconfigurable system operation in the presence of system uncertainties, component failures, and adversarial attacks. In addition, we will consider cyber-physical systems with communication dropouts and time delays. Furthermore, for the connected autonomous vehicle platoon problem we will address network communication attacks as well as incorporate learning mechanisms with attackers and drivers having different levels of rationality. To account for human driver delays, car following models with reaction time delay will also be addressed. Finally, adaptive control and learning architectures for nonlinear models will also be considered.

One of the main challenges in multiagent network systems is dealing with inaccurate sensor data. Specifically, for a group of agents the measurement of the exact location of the other agents relative to a particular agent is often inaccurate due to sensor uncertainty or detrimental environmental conditions. The results in this dissertation can be extended to develop several fundamental results on set-valued protocols for almost consensus of multiagent systems with uncertain interagent communication [123]. A wide variety of ideas can be explored, including set-valued invariance and Lyapunov theorems for discrete- and continuous-time multiagent system stabilization and optimality with switching graph topologies and hierarchical control architectures.

Since communication links among multiagent systems are often unreliable due to multipath effects and exogenous disturbances, the information exchange topologies in network systems are often dynamic. In particular, link failures or creations in network multiagent systems result in switchings of the communication topology. This is the case, for example, if information between agents is exchanged by means of line-of-sight sensors that experience periodic communication dropouts due to agent motion. Variation in network topology introduces control input discontinuities, which in turn give rise to discontinuous dynamical systems. In addition, the communication topology may be time-varying.

In current research, we are developing a unified framework for addressing consensus, flocking, and cyclic pursuit problems for multiagent dynamical systems with fixed and dynamic graph topologies. The proposed framework involves a novel class of fixed-order, energy-based hybrid controllers as a means for achieving cooperative control formations. These dynamic controllers combine a logical switching architecture with continuous dynamics leading to a hybrid closed-loop system described by impulsive differential equations [52], and addresses general nonlinear dynamical systems without limiting consensus and formation control protocols to single and double integrator models.

As discussed in Chapter 8, dynamical network systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles, microsatellite clusters, mobile robotics, autonomous underwater vehicles, distributed sensor networks, power grid systems, and congestion control in communication networks. A major future research thrust would be to develop stochastic optimal control algorithms to address finite-time stabilization, nonlinear fixed-architecture dynamic control, universal feedback stabilizers, and robust stabilization involving both stochastic and deterministic uncertainty as well as averaged and worst-case performance criteria. This will allow us to address finite-time coordination between agents, model uncertainty, and exogenous disturbances.

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